UNIVERSAL ASYMPTOTICS FOR POSITIVE CATALYTIC FUNCTIONAL EQUATIONS

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One Functional Equation

(Folkore) Theorem [Bender, Canfield, Meir+Moon, ...]

Suppose that $\boxed{\Phi(z,y)}$ is a polynomial with $\Phi(0,0) = 0$ and **non-negative coefficients** that depends on z and is **non-linear** in y.

Then the power series solution $y(z) = \sum y_n z^n$ of the functional equation

$$\boxed{y(z) = \Phi(z, y(z))}$$

satisfies (for some constants $c, \gamma > 0$)

$$\boxed{y_n = [z^n]\, y(z) \sim c \cdot n^{-3/2} \gamma^n.}, \qquad n \equiv n_0 \bmod d,$$

and $y_n = 0$ for $n \not\equiv n_0 \bmod d$, where $d \geq 1$ is the *period of the equation*.

One Functional Equation

Binary Trees. $B(z) = \sum_{n \geq 0} B_n z^n$

$$\boxed{B(z) = z(1 + B(z)^2)}$$

$$\boxed{B_n \sim c \cdot n^{-3/2} 2^n}, \qquad n \equiv 1 \bmod 2.$$

One Functional Equation

Squareroot Singularity

The asymptotic expansion

$$y_n = [z^n]\, y(z) \sim c \cdot n^{-3/2} \gamma^n$$

is related to the **universal squareroot singularity**

$$\boxed{y(z) = g(z) - h(z)\sqrt{1 - \frac{z}{z_0}}}$$

of the solution of $y(z) = \Phi(z, y(z))$.

In particular we have $\gamma = 1/z_0$.

One Functional Equation

Strongly connected positive systems

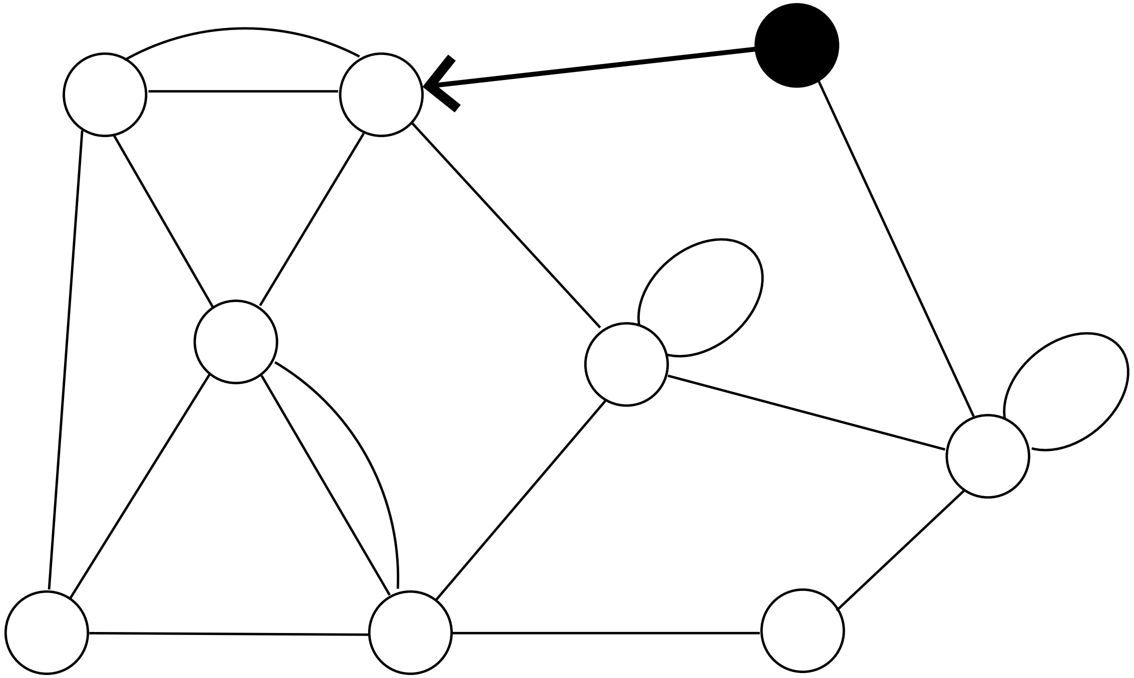
The same property holds for **strongly connected positive polynomial** systems:

$$
\begin{aligned}
y_1 &= \Phi_1(z, y_1, \ldots, y_k), \\
&\vdots \\
y_k &= \Phi_k(z, y_1, \ldots, y_k).
\end{aligned}
$$

For all $j = 1, \ldots, k$ we have

$$\boxed{y_{j,n} = [z^n]\, y_j(z) \sim c_j \cdot n^{-3/2} \gamma^n}, \qquad n \equiv n_j \bmod d,$$

Planar Maps



A **planar map** is a connected planar graph, possibly with loops and multiple edges, together with an embedding in the plane.
A map is **rooted** if a vertex v and an edge e incident with v are distinguished, and are called the root-vertex and root-edge, respectively. The face to the right of e is called the root-face and is usually taken as the **outer face**.

Planar Maps

Generating functions

$M_{n,k}$ ... number of maps with n edges and outer-face-valency k

$$M(z,u) = \sum_{n,k} M_{n,k} u^k z^n$$

$$\boxed{M(z,u) = 1 + zu^2 M(z,u)^2 + uz \frac{uM(z,u) - M(z,1)}{u-1}.}$$

u ... **"catalytic variable"**

Planar Maps

$M_n = \sum_k M_{n,k}$ ... number of rooted maps with n edges [Tutte]

$$\boxed{M_n = \frac{2(2n)!}{(n+2)!n!}3^n}$$

The proof is given with the help of generating functions and the so-called **quadratic method**.

Asymptotics:

$$\boxed{M_n \sim c \cdot n^{-5/2} 12^n}$$

The Quadratic Method

Completing the square leads to

$$\boxed{\left[2u^2z(1-u)M(z,u)-(1-u+u^2z)\right]^2 = H(z,u,M(z,1))}$$

with

$$H(z,u,y) = 4(u-1)u^3z^2y + u^4z^2 - 4u^4z + 6u^3z - 2u^2z + u^2 - 2u + 1.$$

General Form:

$$\boxed{[G_1(z,u,y(z))M(z,u) + G_2(z,u,y(z))]^2 = H(z,u,y(z))}$$

where $y(z)$ abbreviates $M(z,1)$.

The Quadratic Method

$$[G_1(z,u,y(z))M(z,u) + G_2(z,u,y(z))]^2 = H(z,u,y(z))$$

Key observation:

$$\boxed{\exists\, u(z) \text{ with } H(z,u(z),y(z)) = 0 \quad \Longrightarrow \quad H_u(z,u(z),y(z)) = 0}$$

Quadratic Method

1. Solve the system $\boxed{H(z,u(z),y(z)) = 0, \quad H_u(z,u(z),y(z)) = 0}$
2. $M(z,1) = y(z)$
3. $M(z,u) = \left(\sqrt{H(z,u,y(z))} - G_2(z,u,y(z))\right) / G_1(z,u,y(z))$.

The Quadratic Method

Planar Maps

ad 1. $u = u(z)$ and $y(z) = M(z,1)$ are determined by

$$z = \frac{(1-u)(2u-3)}{u^2}, \quad M(z,1) = -u\frac{3u-4}{(2u-3)^2}$$

ad 2. Elimination gives an equation for $M = M(z,1)$:

$$27z^2M^2 - 18zM + M + 16z - 1 = 0$$

and consequently

$$M(z,1) = -\frac{1}{54z^2}\left(1 - 18z - \boxed{(1-12z)^{3/2}}\right) = \sum_{n\ge 0} \frac{2(2n)!}{(n+2)!n!}3^n z^n.$$

ad 3.

$$M(z,u) = \frac{\sqrt{H(z,u,M(z,1))} + 1 - u + u^2 z}{2u^2 z(1-u)}.$$

The Quadratic Method

Planar Maps

The singular behavior

$$M(z,1) = -\frac{1}{54z^2}\left(1 - 18z - \boxed{(1-12z)^{3/2}}\right)$$

of the form $(1-12z)^{3/2}$ is directely related to the asymptotic expansion

$$\boxed{M_n \sim \frac{2}{\sqrt{\pi}} \cdot n^{-5/2} 12^n.}$$

Universal Asymptotics for Catalytic Equations

THEOREM

Suppose that $\boxed{Q(y_0, y_1, z, v)}$ is a polynomial with **non-negative coefficients** that is **non-linear** in y_0, y_1 (and depends on y_0, y_1) and $F(v)$ a non-negative polynomial in v.

Then the power series solution $M(z,v) = \sum M_{n,k} z^n v^k$ of the functional equation

$$\boxed{M(z,v) = F(v) + zQ\left(M(z,v), \frac{M(z,v) - M(z,0)}{v}, z, v\right)}$$

satisfies (for some constants $c, \gamma > 0$)

$$\boxed{M_n = \sum_k M_{n,k} = [z^n]\, M(z,0) \sim c \cdot n^{-5/2} \gamma^n.}, \qquad n \equiv n_0 \bmod d,$$

and $M_n = 0$ for $n \not\equiv n_0 \bmod d$, where $d \geq 1$ is the *period of the equation*.

Universal Asymptotics for Catalytic Equations

Algebraic function [Bousquet-Melou+Jehanne]

For all polynomials $Q(y_0, y_1, z, v)$ and $F(u)$ there exists a unique power series solution $M(z,v)$ of the equation

$$M(z,v) = F(v) + zQ\left(M(z,v), \frac{M(z,v) - M(z,0)}{v}, z, v\right)$$

and the function $M(z,v)$ is **algebraic**.

Universal Asymptotics for Catalytic Equations

Planar Maps

$$M(z,u) = 1 + zu^2 M(z,u)^2 + uz\frac{uM(z,u) - M(z,1)}{u-1}.$$

With $u = v+1$ and $\tilde{M}(z,v) = M(z,v+1)$ we get

$$\boxed{\tilde{M}(z,v) = 1 + z(v+1)\left((v+1)\tilde{M}(z,v)^2 + \tilde{M}(z,v) + \frac{\tilde{M}(z,v) - \tilde{M}(z,0)}{v}\right)}.$$

$$F(v) = 1, \quad Q(y_0, y_1, z, v) = y_0^2(v+1)^2 + y_0(v+1) + y_1(v+1).$$

Universal Asymptotics for Catalytic Equations

Bipartite Planar Maps (or Eulerian planar maps by duality)

$$E(z,u) = 1 + zu^2 E(z,u)^2 + u^2 z \frac{E(z,u) - E(z,1)}{u^2 - 1}.$$

With $u^2 = v + 1$ and $\tilde{E}(z,v) = E(z, \sqrt{1+v})$ we get

$$\boxed{\tilde{E}(z,v) = 1 + z(v+1)\tilde{E}(z,v)^2 + (v+1)z \frac{\tilde{E}(z,v) - \tilde{E}(z,0)}{v}}.$$

$$F(v) = 1, \quad Q(y_0, y_1, z, v) = y_0^2(v+1) + y_1(v+1).$$

Universal Asymptotics for Catalytic Equations

2-Connected Planar Maps

$$B(z,u) = z^2u + zuB(z,u) + u(z + B(z,u))\frac{B(z,u) - B(z,1)}{u-1}.$$

With $u = v + 1$ and $\tilde{B}(z,v) = B(z,v+1)$ we get

$$\boxed{\tilde{B}(z,v) = z^2(v+1) + z(v+1)\tilde{B}(z,v) + (v+1)(z + \tilde{B}(z,v))\frac{\tilde{B}(z,v) - \tilde{B}(z,0)}{v}.}$$

This is not precisely of the above type but it also works.

Universal Asymptotics for Catalytic Equations

Planar Triangulations

$$T(z,u) = (1-uT(z,u)) + (z+u)T(z,u)^2 + z(1-uT(z,u))\frac{T(z,u)-T(z,0)}{u}.$$

With $u = v$ and $\tilde{T}(z,v) = T(z,u)/(1-uT(z,u))$ we get

$$\boxed{\tilde{T}(z,v) = 1 + v\tilde{T}(z,v) + z(1+\tilde{T}(z,v))\frac{\tilde{T}(z,v)-\tilde{T}(z,0)}{v}.}.$$

This is (again) not precisely of the above type but it also works.

Two Ways of Seeing the Quadratic Method

Bousquet-Melou+Jehanne - Approach

Let $P(x_0, x_1, z, v)$ be an analytic function such that $(y(z) = M(z, 0))$

$$\boxed{P(M(z,v), y(z), z, v) = 0.}$$

By taking the derivative with respect to v we get

$$P_{x_0}(M(z,v), y(z), z, v)\, M_v(z,v) + P_v(M(z,v), y(z), z, v) = 0.$$

Key obervation:

$$\boxed{\exists\, v(z) : P_v(M(z,v(z)), y(z), z, v(z)) = 0 \Longrightarrow P_{x_0}(M(z,v(z)), y(z), z, v(z)) = 0}$$

Thus, with $f(z) = M(z, v(z))$ we get the system for $f(z), y(z), u(z)$

$$\begin{aligned} P(f(z), y(z), z, v(z)) &= 0 \\ P_{x_0}(f(z), y(z), z, v(z)) &= 0 \\ P_v(f(z), y(z), z, v(z)) &= 0. \end{aligned}$$

Remark: For the quadratic case this just reduces to the quadratic method.

Two Ways of Seeing the Quadratic Method

Bousquet-Melou+Jehanne - Approach

Set (as given in our case)

$$\boxed{P(x_0, x_1, z, v) = F(v) + zQ(x_0, (x_0 - x_1)/v, z, v) - x_0.}$$

Then the system $P = 0$, $P_{x_0} = 0$, $P_v = 0$ rewrites to

$$\begin{aligned}
f(z) &= F(v(z)) + zQ(f(z), w(z), z, v(z)),\\
v(z) &= zv(z)Q_{y_0}(f(z), w(z), z, v(z)) + zQ_{y_1}(f(z), w(z), z, v(z)),\\
w(z) &= F_v(v(z)) + zQ_v(f(z), w(z), z, v(z)) + zw(z)Q_{y_0}(f(z), w(z), z, v(z)),
\end{aligned}$$

where

$$w(z) = \frac{f(z) - y(z)}{v(z)}.$$

This is a **positive strongly connected polynomial system**.

Two Ways of Seeing the Quadratic Method

Bousquet-Melou+Jehanne - Approach

Thus, by the **Folklore Theorem for Systems** the solution functions $f(z), v(z), w(z)$ have a **squareroot singularity** at some common singularity z_0:

$$f(z) = g_1(z) - h_1(z)\sqrt{1 - \frac{z}{z_0}},$$
$$v(z) = g_2(z) - h_2(z)\sqrt{1 - \frac{z}{z_0}},$$
$$w(z) = g_3(z) - h_3(z)\sqrt{1 - \frac{z}{z_0}}.$$

$\Longrightarrow$ $y(z) = f(z) - v(z)w(z)$ has also a squareroot singularity at z_0

$$y(z) = g_4(z) - h_4(z)\sqrt{1 - \frac{z}{z_0}} = a_0 + a_1\sqrt{1 - \frac{z}{z_0}} + a_2\left(1 - \frac{z}{z_0}\right) + a_3\left(1 - \frac{z}{z_0}\right)^{3/2} + \cdots$$

but maybe there are **cancellations of coefficients** a_j (and actually this happens!!!).

Two Ways of Seeing the Quadratic Method

Weierstrass Preparation Theorem - Approach

We have $\boxed{P = P_{y_0} = 0}$ for $y_0 = f(z_0), y_1 = y(z_0), z = z_0, v = v(z_0)$. Hence the **Weierstrass Preparation Theorem** implies that P can be represented by

$$\boxed{P(y_0, y_1, z, v) = K(y_0, y_1, z, v)\left((y_0 - G(y_1, z, v))^2 - H(y_1, z, v)\right)}$$

with analytic function K, G, H with $K \neq 0$.

The relations $P = 0$, $P_{x_0} = 0$, $P_v = 0$ imply

$$\boxed{H(y(z), z, v(z)) = 0, \quad H_v(y(z), z, v(z)) = 0}$$

Furthermore for the **critical point** $y_1 = y(z_0), z = z_0, v = v(z_0)$ we also have

$$\boxed{H_{vv}(y(z_0), z_0, v(z_0)) = 0.}$$

Two Ways of Seeing the Quadratic Method

Weierstrass Preparation Theorem - Approach

Lemma. Suppose that y_0, z_0, v_0 are complex numbers and that $H(z, v, y)$ is a function that is analytic at (y_0, z_0, v_0) and satisfies the properties

$$\boxed{H(y_0, z_0, v_0) = 0, \quad H_u(y_0, z_0, v_0) = 0, \quad H_{uu}(y_0, z_0, v_0) = 0}$$

and (for $(y, z, v) = (y_0, z_0, v_0)$):

$$H_y \neq 0, \quad H_{vy} \neq 0, \quad H_{vvv} \neq 0, \quad H_z H_{vy} \neq H_y H_{vz}$$

Then the system of functional equations

$$\boxed{H(y(z), z, v(z)) = 0, \quad H_u(y(z), z, v(z)) = 0}$$

has precisely two solutions $v(z)$ and $y(z)$ with $v(z_0) = v_0$ and $y(z_0) = y_0$ which are given by

$$v(z) = \overline{g}_1(z) \pm \overline{h}_(z)\sqrt{1 - \frac{z}{z_0}},$$

$$y(z) = \overline{g}_2(z) \pm \overline{h}_2(z)\left(1 - \frac{z}{z_0}\right)^{3/2}$$

in a neighbourhood of z_0.

Two Ways of Seeing the Quadratic Method

Weierstrass Preparation Theorem - Approach: Proof steps

1. $H_v(z, v, y) = 0 \quad \Longrightarrow \quad y = Y(z, v)$

2. $H(z, v, Y(z, v)) = 0 \quad \Longrightarrow \quad v = v(z)$

3. $y(z) = Y(z, v(z))$

The Analytic Quadratic Method

ad 1. $H_v(z, v, Y(z, v)) = 0$

Implicit function theorem $\Longrightarrow Y(z, v)$ analytic at (z_0, v_0) but

$$\boxed{Y_v(z_0, v_0) = -\frac{H_{vv}}{H_{vy}} = 0.}$$

ad 2. $H(z, v, Y(z, v)) = 0$, $v = v(z)$

Folklore Theorem $\Longrightarrow v(z) = g_1(z) \pm g_2(z)\sqrt{1 - \frac{z}{z_0}}$

Two Ways of Seeing the Quadratic Method

Weierstrass Preparation Theorem - Approach: Proof steps

ad 3. $y(z) = Y(z, v(z))$

$Y_v(z_0, v_0) = 0 \Longrightarrow$

$$\begin{aligned}
y(z) &= Y(z, v(z)) \\
&= y_0 + Y_z(z_0, v_0)(z - z_0) + \frac{1}{2} Y_{vv}(z_0, v_0)(v(z) - v_0)^2 \\
&+ Y_{vz}(z_0, v_0)(z - z_0)(v(z) - v_0) + \frac{1}{6} Y_{vvv}(z_0, v_0)(v(z) - u_0)^3 \\
&+ O((z - z_0)^2) \\
&= h_1(z) \pm h_2(z) \left(1 - \frac{z}{z_0}\right)^{3/2}
\end{aligned}$$

Two Ways of Seeing the Quadratic Method

Summing up

1. $y(z) = M(z,0)$ has a squareroot singularity at z_0 with possible cancellations of coefficients of the singular expansions.

2. $y(z) = \overline{g}_2(z) \pm \overline{h}_2(z) \left(1 - \frac{z}{z_0}\right)^{3/2}$

3. It is possible to check that $\overline{h}_2(z_0) \neq 0$.

4. Thus the dominant singularity of $y(z)$ is of the form $\boxed{(1 - z/z_0)^{3/2}}$ which leads to the (universal) behavior of $M_n = [z^n]M(z,0) \sim cn^{-5/2} z_0^{-n}$.

Extensions

- Systems of (Catalytic) Equations
- Additional counting parameters which lead to central limit theorems (via Hwang's Quasi-Power-Theorem)
- ...

Thank You!