Enumerating Lambda Terms By Weighted Length of Their de Bruijn Representation

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joint work with Olivier Bodini and Zbigniew Gołębiewski

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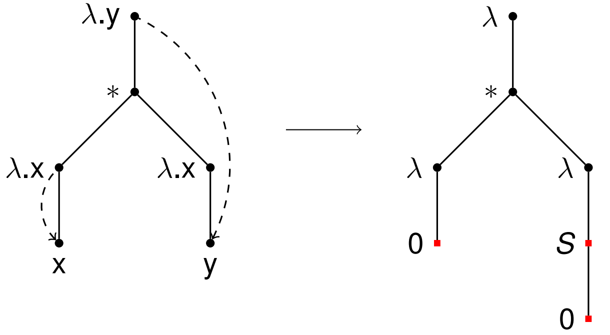
Definition of lambda terms

$$T ::= x \mid \lambda x.T \mid T * T \quad \rightarrow \quad T ::= S^n 0 \mid \lambda T \mid T * T$$

$\lambda x.T$: abstraction, unary node $\qquad (T * T)$: application, binary node

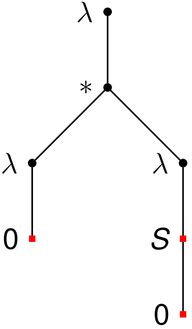
$$\lambda y.((\lambda x.x) * (\lambda x.y)) \rightarrow \lambda(\lambda 1 * \lambda 2) \rightarrow \lambda((\lambda 0) * (\lambda(S0)))$$



m-open lambda terms

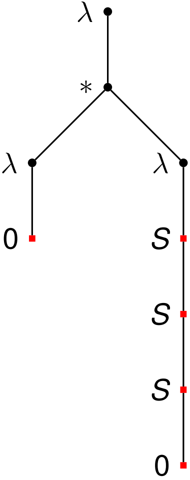
closed lambda term (0-open)

$\lambda((\lambda 0) * (\lambda(S0)))$



2-open lambda term

$\lambda((\lambda 0) * (\lambda(SSS0)))$



General notion of size

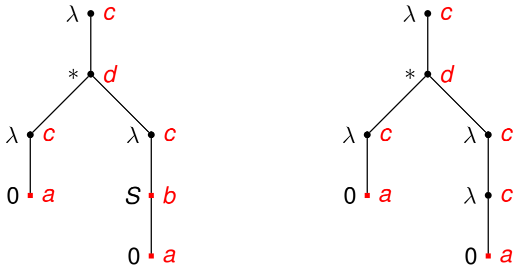
$$\begin{aligned} |0| &= a \\ |S| &= b \\ |\lambda M| &= |M| + c \\ |MN| &= |M| + |N| + d. \end{aligned}$$

Assumptions

1. *a, b, c, d are nonnegative integers,*
2. $a + d \geq 1$,
3. $b, c \geq 1$,
4. $\gcd(b, c, a + d) = 1$.

General notion of size

$$\lambda((\lambda 0) * (\lambda(S0))) \quad \neq \quad \lambda((\lambda 0) * (\lambda(\lambda 0)))$$



size: $2a + b + 3c + d$ size: $2a + 4c + d$

- natural counting (Bendkowski, Grygiel, Lescanne, Zaionc 2015): $a = b = c = d = 1$
- less natural counting (Bendkowski, Grygiel, Lescanne, Zaionc 2015): $a = 0,\ b = c = 1,\ d = 2$
- binary lambda calculus (Tromp 2006): $b = 1,\ a = c = d = 2$

Combinatorial specification and lambda terms

$$\mathcal{L} = \textsc{Seq}(\mathcal{S}) \times \mathcal{Z} \;\cup\; \mathcal{U} \times \mathcal{L} \;\cup\; \mathcal{A} \times \mathcal{L}^2$$

- $\mathcal{L}$ – the class of lambda terms,
- $\mathcal{Z}$ – the class of zeros,
- $\mathcal{S}$ – the class of successors,
- $\mathcal{U}$ – the class of abstractions,
- $\mathcal{A}$ – the class of applications.

Remark: $\mathcal{Z}, \mathcal{S}, \mathcal{U}, \mathcal{A}$ contain only one atomic object.
Thus

$$L(z) = z^a \sum_{j=0}^{\infty} z^{bj} + z^c L(z) + z^d L(z)^2,$$

$[z^n]L(z)$=number of lambda terms of size n.

m-open terms and functional equations

Let

$$\mathcal{L}_m = \textsc{Seq}_{\leq m-1}(\mathcal{S}) \times \mathcal{Z} \;\cup\; \mathcal{U} \times \mathcal{L}_{m+1} \;\cup\; \mathcal{A} \times \mathcal{L}_m^2.$$

- $L_{m,n}$ – the number of m-open lambda terms of size n,
- $L_m(z) = \sum_{n\geq 0} L_{m,n} z^n \quad ([z^n]\, L_m(z) = L_{m,n})$

$$L_m(z) = z^a \sum_{j=0}^{m-1} z^{bj} + z^c L_{m+1}(z) + z^d L_m(z)^2$$

- $L_0(z)$ is the gen. fun. of the set $\mathcal{L}_0$ of closed lambda terms,
- $L_\infty(z)$ is the gen. fun. of the set $\mathcal{L}_\infty = \mathcal{L}$ of all lambda terms.

$L_\infty(z)$ – all terms

Solving $L_\infty(z) = z^a \sum_{j=0}^{\infty} z^{bj} + z^c L_\infty(z) + z^d L_\infty(z)^2$. gives

Proposition

Let $\rho = \mathrm{RootOf}\left\{(1-z^b)(1-z^c)^2 - 4z^{a+d}\right\}$. *Then*

$$L_\infty(z) = a_\infty - b_\infty \sqrt{1 - \frac{z}{\rho}} + \mathrm{O}\left(\left|1 - \frac{z}{\rho}\right|\right),$$

for some constants $a_\infty > 0, b_\infty > 0$ *that depend on* a, b, c, d. *Thus the coefficients of* $L_\infty(z)$ *satisfy*

$$L_{\infty,n} \sim \frac{b_\infty}{2\sqrt{\pi}} \rho^{-n} n^{-3/2}, \text{ as } n \to \infty.$$

Idea: Replace $\mathcal{L}_m$ by pruning de Bruijn indices

We have

$$L_m(z) = z^a \sum_{j=0}^{m-1} z^{bj} + z^c L_{m+1}(z) + z^d L_m(z)^2.$$

$\mathcal{L}_m^{(h)}$ – lambda terms in $\mathcal{L}_m$ where the length of each string of successors is bounded by h

$$L_m^{(h)}(z) = \begin{cases} z^a \sum_{j=0}^{m-1} z^{bj} + z^c L_{m+1}^{(h)}(z) + z^d L_m^{(h)}(z)^2 & \text{if } m < h, \\ z^a \sum_{j=0}^{h-1} z^{bj} + z^c L_h^{(h)}(z) + z^d L_h^{(h)}(z)^2 & \text{if } m \geq h, \end{cases}$$

because for $m \geq h$ we have $L_m^{(h)}(z) = L_h^{(h)}(z)$.
$\rightsquigarrow$ upper and lower bounds.

Theorem (G., Gołębiewski 2016)

Let $\rho = \mathrm{RootOf}\left\{(1-z^b)(1-z^c)^2 - 4z^{a+d}\right\}$*. Then there exist positive constants* $\underline{C}$ *and* $\overline{C}$ *(depending on* a, b, c, d *and* m*) such that the number of* m*-open lambda terms of size* n *satisfies*

$$\liminf_{n\to\infty} \frac{L_{m,n}}{\underline{C} n^{-\frac{3}{2}} \rho^{-n}} \geq 1 \quad \mathit{and} \quad \limsup_{n\to\infty} \frac{L_{m,n}}{\overline{C} n^{-\frac{3}{2}} \rho^{-n}} \leq 1,$$

Remark

In case of given a, b, c, d *and* m *we can compute numerically such constants* $\underline{C}$ *and* $\overline{C}$*.*

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Remark

In case of given a, b, c, d *and* m *we can compute numerically such constants* $\underline{C}$ *and* $\overline{C}$.
For instance, for natural counting we have

$$\underline{C}^{(nat)} \approx 0.0779099{\color{red}5266}\ldots,$$
$$\overline{C}^{(nat)} \approx 0.0779099{\color{red}8229}\ldots.$$

Theorem

Let $\rho = \mathrm{RootOf}\left\{(1-z^b)(1-z^c)^2 - 4z^{a+d}\right\}$*. Then there exists a positive constant C (depending on* a, b, c, d *and m) such that the number of m-open lambda terms of size n satisfies*

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Replace $L_m(z)$ by $L_\infty(z)$ and trace back:

$$a_m := a_\infty, \qquad b_m := b_\infty; \qquad L_{m,m}(z) := L_\infty(z).$$

$$L_{m,m}(z) = L_\infty(z) = a_\infty - b_\infty\sqrt{1 - \frac{z}{\rho}},$$

$$L_{i,m}(z) = z^a \sum_{j=0}^{i-1} z^{jb} + z^c L_{i+1,m}(z) + z^d L_{i,m}(z)^2.$$

Then, eventually we obtain

$$L_{0,m}(z) = a_{0,m} - b_{0,m}\sqrt{1 - \frac{z}{\rho}}$$

$$L_{0,m}(z) = a_{0,m} - b_{0,m}\sqrt{1 - \frac{z}{\rho}}$$

Lemma

The sequences $(a_{0,m})_{m\geq 0}$ and $(b_{0,m})_{m\geq 0}$ are convergent.

Proof.

We know that $\lim_{m\to\infty} L_{0,m}(z) = L_0(z)$, uniformly in $[0,\rho]$ and that $L_{0,m}(z)$ is decreasing. Thus

$$a_{0,m} = L_{0,m}(\rho) \longrightarrow L_0(\rho) =: a_0, \text{ as } m \to \infty.$$

b_0, m is increasing and bounded by b_∞, thus converges to b_0. $\square$

The theorem follows now from the uniform convergence of $L_{0,m}(z)$ and the local shape of these functions.

Boltzmann sampling

Singular Boltzmann output size according to Boltzmann distribution

$$\mathbb{P}\{N = n\} = \frac{a_n \rho^n}{A(\rho)}$$

where $A(\rho)$ is the generating function and ρ its dominant singularity.

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The sampler: Construct superclass $\mathcal{L}_{N,0} \supseteq \mathcal{L}_0$ tending to $\mathcal{L}_0$ and reject unwanted results:

$$\begin{cases} L_{N,0} & = zL_{N,1} + zL_{N,0}^2, \\ L_{N,1} & = zL_{N,2} + zL_{N,1}^2 + z, \\ L_{N,2} & = zL_{N,3} + zL_{N,2}^2 + z + z^2 \quad , \\ \dots & = \dots, \\ L_{N,N-1} & = zL_{N,N} + zL_{N,N-1}^2 + z\dfrac{1 - z^{N-1}}{1 - z}, \\ L_{N,N} & = zL_{N,N} + zL_{N,N}^2 + \dfrac{z}{1 - z} \end{cases}$$

Rejection if de Bruijn index larger than N drawn in ΓL_N.

Boltzmann sampling – Costs

Without extra rejection:

- traditional framework of singular Boltzmann sampling
- linear if target size is in $((1-\varepsilon)n, (1+\varepsilon)n)$

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Extra rejection:

- Costs bounded by object size
- Unwanted terms are open terms in $\mathcal{L}_{0,N}$:

$$\frac{[z^n]L_0(z)}{[z^n]L_{N,0}(z)} \longrightarrow 1.$$

Speed is exponential: For $N = 20$, the proportion of closed terms is 0.999999998.

Boltzmann sampling – Experiments

Experiments with $N = 20$:

$\mathbb{P}\{\text{unary}\} \approx 0.2955977425, \qquad \mathbb{P}\{\text{binary}\} = \mathbb{P}\{\text{leaf}\} \approx 0.3522011287.$

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Average number of leaves in a term of size n is asymptotically $0.3522011287 \cdots n$.
$X_N := \#$ leaves in $\mathcal{L}_{N,0} \stackrel{d}{=} \#$ leaves in $\mathcal{L}_{N,N}$.

Thus, all moments of X_N and $X := \#$ leaves in a closed term are asymptotically equal.

Theorem

Let X_n the number of variables in a lambda-term of size n. Then

$$X_n \sim \mathcal{N}(\mu n, \sigma^2 n)$$

where μ and σ^2 tend to $\frac{1-\rho}{2} \approx 0.3522011287$.

Boltzmann sampling – Experiments

Sampler has same complexity as sampler for trees (linear in approximate size)

On laptop with CPU i7-5600U, clock rate 2.6 GHz, it is possible to draw a lambda-term of size in the range [1 000 000, 2 000 000] in less than 10 minutes.

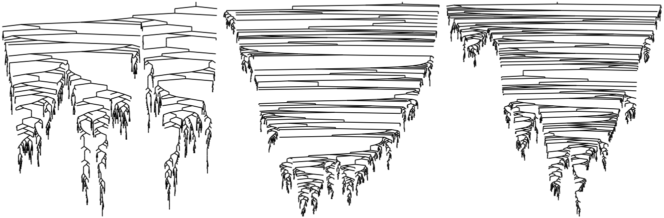

Figure: Three uniform random lambda-terms of size 2098, 2541, 2761.

Related work and perspectives

- Study further properties of these terms
- terms of bounded unary height
  cf. Bodini, Gardy, G. 2011 and Bodini, Gardy, G, Gołębiewski 2016
  - Shape characteristics of terms with bounded unary height (G., Larcher, in progress)
- restricting the number of variables bound by an abstraction
  cf. Bodini, Gardy, Jacquot 2010 (BCI/BCK), Bodini, Gardy, G., Jacquot 2013(gen. BCI), Bodini, G. 2014 (BCK_2)
  - Shape characteristics of BCI/BCK/gen BCI terms (G., Larcher, in progress)

Thank you!