

Asymptotic problems arising in the classification
of symbolic dynamical systems...

...or when are two Markov chains
magic word isomorphic?

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Dynamical Systems

B O R E L S Y S T E M S

M A R K O V C H A I N S

Positive Recurrent
(PR)

Null Recurrent
(N)

Transient
(T)

Exponentially
Recurrent
(ER)

L O O P S Y S T E M S

Borel Systems

(X, S) Borel system

- (X, \mathcal{X}) is a standard Borel space:
 - X is a complete separable metric space.
 - \mathcal{X} is the Borel σ -algebra.
- $S: X \rightarrow X$ is a Borel automorphism:
 - S is a bimeasurable bijection, i.e.

$$S^{-1}\mathcal{X} := \{S^{-1} : E \in \mathcal{X}\} = S\mathcal{X} = \mathcal{X}.$$

$\mathbb{P}(X) = \{\mu: \mathcal{X} \rightarrow [0, 1] \text{ invariant (Borel) probability measure}\}$

Periodic points and dynamic zeta function

(X, S) Borel system

$$\mathcal{P}_n := \{x \in X \mid S^n(x) = x\} \quad P_n := |\mathcal{P}_n|$$

$$\mathcal{P} = \bigcup_{n \geq 1} \mathcal{P}_n \quad \text{Free}(X) := X \setminus \mathcal{P}(X)$$

$$\text{Period} := \gcd\{n \mid P_n \neq 0\} = 1$$

$$\zeta(z) = \exp \left(\sum_{n \geq 1} \frac{P_n}{n} z^n \right)$$

Markov chains

markov chains

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Markov State Diagram

Figure 2

State Transition Matrix:

$$P = \begin{bmatrix} P_{AA} & P_{AB} & P_{AC} \\ P_{BA} & P_{BB} & P_{BC} \\ P_{CA} & P_{CB} & P_{CC} \end{bmatrix}$$

Markov Chain Diagrams:

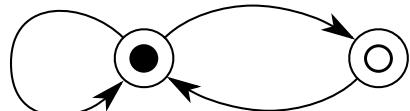
- Bull market, Bear market, Stagnant market
- E to A
- Cloudy, Rain, Sunny
- A to B
- s₀ to s₇
- A, B, C states with transitions between them
- 0 to 5 states with transitions between them
- sunny, cloudy, rainy
- 1, 2, 3 states with transitions between them
- Cheerful, So-so, Sad
- Large complex grid of states with many transitions

Symbolic dynamical systems

(Σ, σ) Markov shift

Strongly connected digraph

$$G = (V, E)$$



Adjacency matrix

$$A = \begin{pmatrix} \bullet & \circ \\ \bullet & \circ \end{pmatrix} \in \mathcal{M}_{V \times V}\{0, 1\}$$

$$\Sigma := \{\mathbf{x} = (x_n)_{n \in \mathbb{Z}} \in V^{\mathbb{Z}} \mid A(x_n, x_{n+1}) = 1 \quad \forall n \in \mathbb{Z}\}$$

$$\sigma: \Sigma \rightarrow \Sigma \quad \sigma(\mathbf{x})_n = x_{n+1} \quad \forall n \in \mathbb{Z}$$

CANTOR

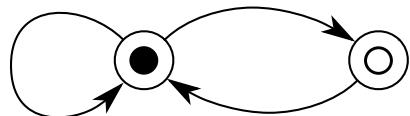


Symbolic dynamical systems

(Σ, σ) Markov shift

Strongly connected digraph

$$G = (V, E)$$
$$E \subset V^2$$



Adjacency matrix

$$A = \begin{pmatrix} \bullet & \circ \\ \bullet & \left(\begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix} \right) \end{pmatrix} \in \mathcal{M}_{V \times V}\{0, 1\}$$

$$\zeta(z) = \frac{1}{\det(I - zA)} = \frac{1}{1 - z - z^2}$$

$\lambda :=$ Spectral radius of A

Entropy: $h(X) := \log \lambda$

Radius of convergence of $\zeta(z)$ is $\frac{1}{\lambda}$

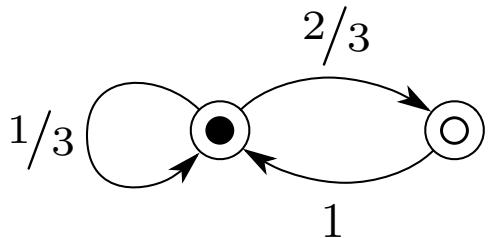
Symbolic dynamical systems

(Σ, σ, μ) Markov chain

Strongly connected digraph

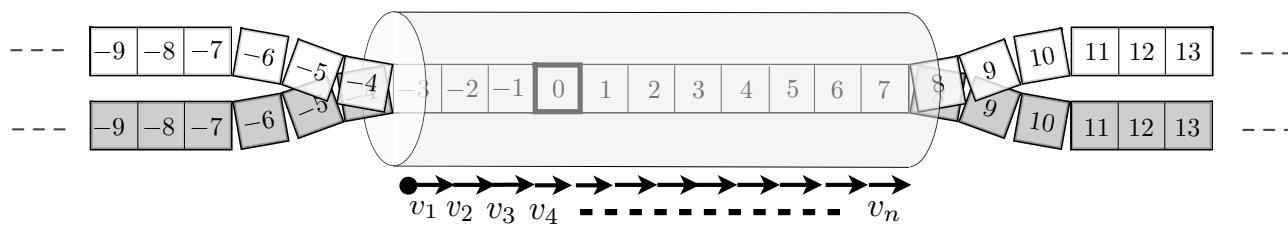
$$G = (V, E)$$

$$E \subset V^2$$



Adjacency matrix

$$A = \begin{pmatrix} \bullet & \circ \\ \bullet & \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix} \in \mathcal{M}_{V \times V} \{0, 1\}$$



Transition matrix

$$P = \begin{pmatrix} \bullet & \circ \\ \bullet & \begin{pmatrix} 1/3 & 2/3 \\ 1 & 0 \end{pmatrix} \end{pmatrix}$$

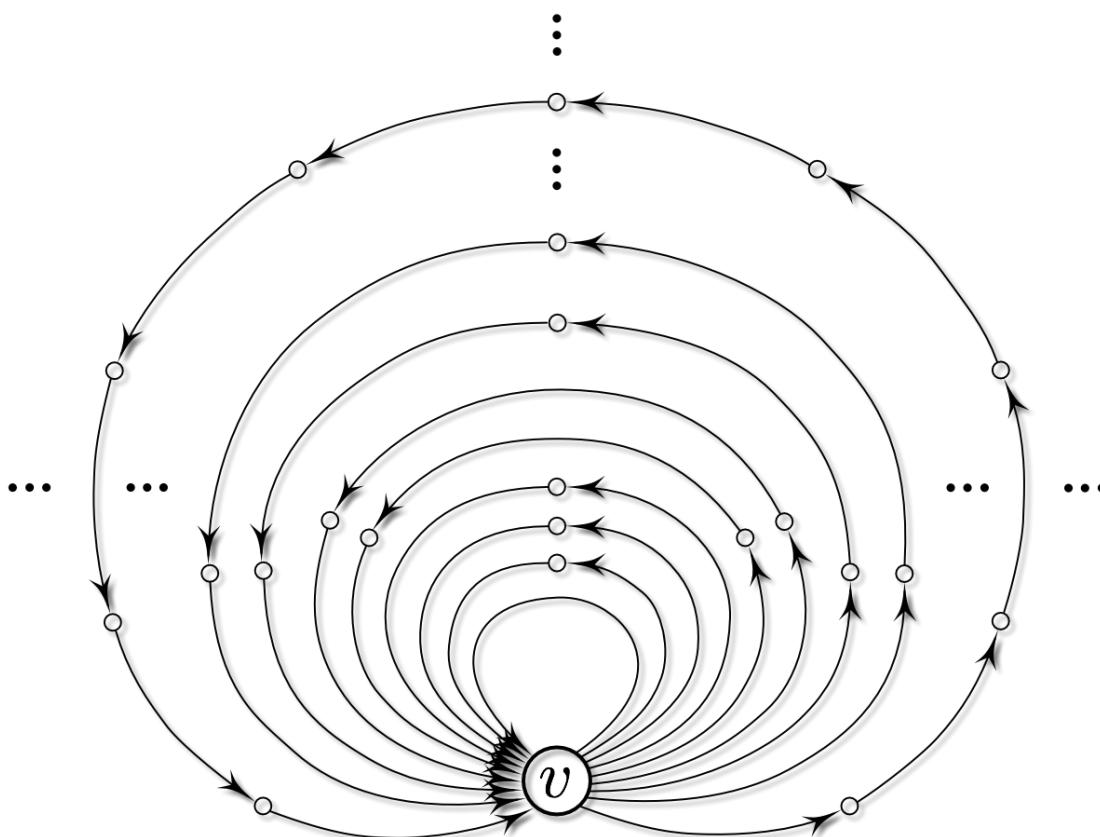
$$\mu([v_1 v_2 \dots v_n]) = \pi_{v_1} P_{v_1, v_2} P_{v_2, v_3} \dots P_{v_{n-1}, v_n}$$

Stationary distribution

$$\pi P = \pi$$

Symbolic dynamical systems

Loop systems



$$f(z) = \sum_{n=1}^{\infty} f_n z^n$$

$$\zeta_f(z) = \frac{1}{1 - f(z)}$$

Classification

(X, S) \longleftrightarrow Borel systems \longrightarrow (Y, T)

Homomorphisms

- $\phi: X \rightarrow Y$
- $\phi \circ S = T \circ \phi$

$$\begin{array}{ccc} X & \xrightarrow{S} & X \\ \phi \downarrow & \circlearrowleft & \downarrow \phi \\ Y & \xrightarrow[T]{} & Y \end{array}$$

Classification

(X, S) \longleftrightarrow Borel systems \longrightarrow (Y, T)

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- | | |
|---|--|
| <ul style="list-style-type: none">• Injective (Embeddings)• Surjective (Factors)• Bijective (Isomorphisms) | <ul style="list-style-type: none">• Defined on “full sets” (almost Borel)• Measurable (Borel)• Continuous a.e. (Finitary)• Continuous• Etc. |
|---|--|

Classification

(X, S)  Borel systems  (Y, T)

Homomorphisms

- $\phi: X \rightarrow Y$
- $\phi \circ S = T \circ \phi$

$A \subset X$ is full if
it is invariant and
 $\mu(A) = 1 \quad \forall \mu \in \mathbb{P}(X)$

- Injective (**Embeddings**)
- Surjective (**Factors**)
- Bijective (**Isomorphisms**)

- Defined on “full sets” (**almost Borel**)
- Measurable (**Borel**)
- Continuous a.e. (**Finitary**)
- Continuous (**Conjugacy**)
- Etc.

Notation

	Continuos	Borel	Almost Borel
Embedding	\hookrightarrow	$\mathbb{B} \hookrightarrow$	$a\mathbb{B} \hookrightarrow$
Factor	\twoheadrightarrow	$\mathbb{B} \twoheadrightarrow$	$a\mathbb{B} \twoheadrightarrow$
Isomorphism	\approx	$\mathbb{B} \approx$	$a\mathbb{B} \approx$

Almost Borel universality of Markov shifts

Theorem (Hochman, 2013) Markov shifts are **almost Borel universal**:

- (X, S) Borel system
 - (Σ, σ) aperiodic Markov shift
 - $h(X, \mu) < h(\Sigma) \quad \forall \mu \in \mathbb{P}(X)$
- $\Rightarrow X \xrightarrow{a\mathbb{B}} \Sigma$

Almost Borel universality of Markov shifts

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-
- The diagram consists of three boxes: an orange box containing $h(X, \mu)$, a red box containing $h(\Sigma)$, and a blue box containing the text "Topological entropy (Borel entropy)" and "Measure theoretical entropy (Kolmogorov-Sinai entropy)". An orange arrow points from the orange box to the blue box. A red arrow points from the red box to the blue box. A thick orange arrow points directly from the bottom of the orange box to the bottom of the blue box.
- Topological entropy
(Borel entropy)
- Measure theoretical entropy
(Kolmogorov-Sinai entropy)

Almost Borel universality of Markov shifts

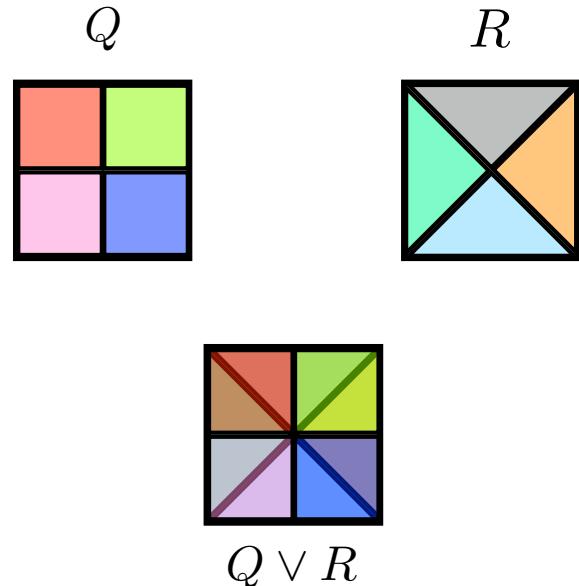
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 - (Σ, σ) aperiodic Markov shift
 - $h(X, \mu) < h(\Sigma) \quad \forall \mu \in \mathbb{P}(X)$
- $\Rightarrow X \xrightarrow{a\mathbb{B}} \Sigma$

$$\boxed{H(Q) = - \sum_{m=1}^k \mu(Q_m) \log \mu(Q_m)}$$

$$\boxed{h_\mu(X, Q) = \lim_{N \rightarrow \infty} \frac{1}{N} H \left(\bigvee_{n=0}^N S^{-n} Q \right)}$$

$$h(X, \mu) = \sup_{Q \text{ finite measurable partition}} h_\mu(X, Q)$$



Almost Borel universality of Markov shifts

Theorem (Hochman, 2013) Markov shifts are almost Borel universal:

- $$\left. \begin{array}{l} \bullet \text{ } (X, S) \text{ Borel system} \\ \bullet \text{ } (\Sigma, \sigma) \text{ aperiodic Markov shift} \\ \bullet \text{ } h(X, \mu) < h(\Sigma) \quad \forall \mu \in \mathbb{P}(X) \end{array} \right\} \Rightarrow X \stackrel{a\mathbb{B}}{\hookrightarrow} \Sigma$$

$$H(Q) = - \sum_{m=1}^k \mu(Q_m) \log \mu(Q_m)$$

$$h_\mu(X, Q) = \lim_{N \rightarrow \infty} \frac{1}{N} H \left(\bigvee_{n=0}^N S^{-n} Q \right)$$

$$h(X, \mu) = \sup_{Q \text{ finite measurable partition}} h_\mu(X, Q)$$

Borel

Kolmogorov-Sinai

$$h(X) = \sup_{\mu \in \mathbb{P}(X)} h(X, \mu)$$

Almost Borel classification of Markov shifts

Theorem (Hochman, 2013) X and Y aperiodic Markov shifts:

$$X \stackrel{a\mathbb{B}}{\approx} Y$$

- $h(X) = h(Y)$
 - $\exists \mu_{\max}(X) \iff \exists \mu_{\max}(Y)$

$$H(Q) = - \sum_{m=1}^k \mu(Q_m) \log \mu(Q_m)$$

$$h_\mu(X, Q) = \lim_{N \rightarrow \infty} \frac{1}{N} H \left(\bigvee_{n=0}^N S^{-n} Q \right)$$

$$h(X, \mu) = \sup_{Q \text{ finite measurable partition}} h_\mu(X, Q)$$

Borel

Kolmogorov-Sinai

$$h(X) = \sup_{\mu \in \mathbb{P}(X)} h(X, \mu)$$

Almost Borel vs. Borel classification for Markov shifts

Theorem (Hochman, 2013) X and Y aperiodic Markov shifts:

$$X \stackrel{a\mathbb{B}}{\approx} Y \iff$$

- $h(X) = h(Y)$
- $\exists \mu_{\max}(X) \iff \exists \mu_{\max}(Y)$

Theorem (Boyle, Buzzi, G. 2014)

X and Y aperiodic and exponentially recurrent Markov shifts:

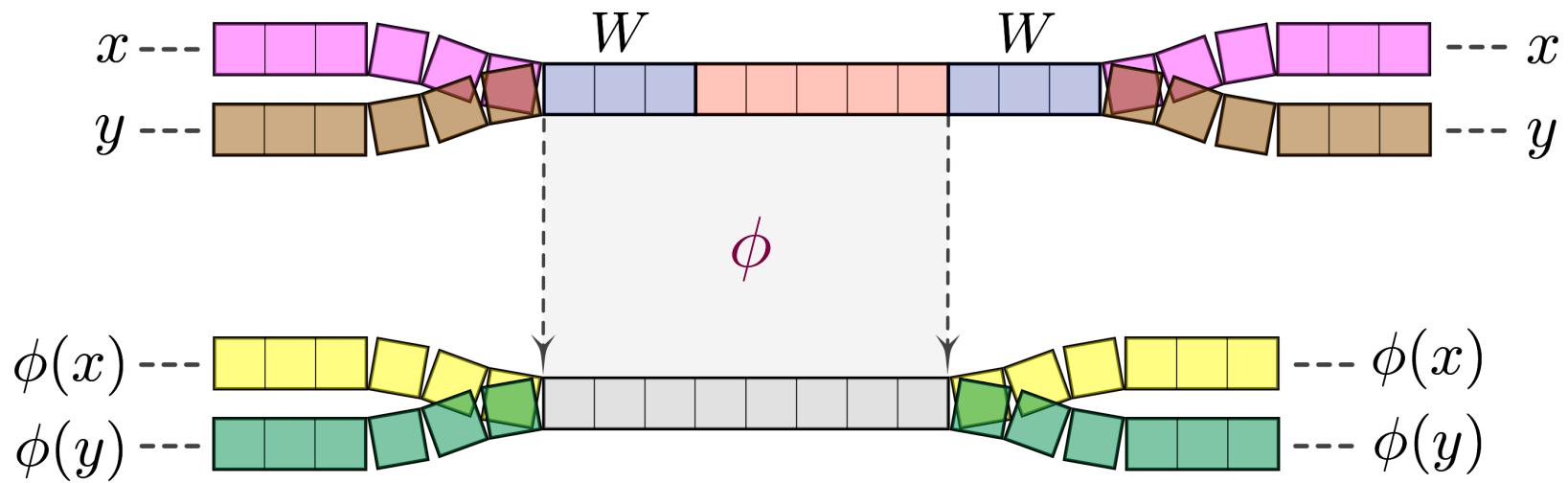
$$\text{Free}(X) \stackrel{\mathbb{B}}{\approx} \text{Free}(Y) \iff h(X) = h(Y)$$

MAGIC WORD ISOMORPHISMS

Magic word isomorphisms

(Σ_A, μ) Markov chains (Σ_B, ν)

$$\phi: \Sigma_A \rightarrow \Sigma_B$$



- Both ϕ and ϕ^{-1} have magic words
- $\phi\mu = \nu$ and $\phi^{-1}\nu = \mu$

Magic word isomorphisms

$$(\Sigma_A, \mu) \quad \longleftrightarrow \quad \text{Markov chains} \quad \longrightarrow \quad (\Sigma_B, \nu)$$

$$\Sigma_A = \{\text{C programs}\} \xrightarrow[\text{Compiler}]{\phi} \Sigma_B = \{\text{Machine Language}\}$$

$\mathbf{x} =$

```
#include <stdio.h>

int main()
{
    int i;
    int a[10];
    printf("Enter student's scores: \n");
    for(i = 0; i < 10; i++) {
        scanf("%d", &a[i]);
    }
    printf("Your student's scores are: \n\n");
    for(i = 0; i < 10; i++) {
        printf("%d\n", a[i]);
    }
    return 0;
}
```

$\phi(\mathbf{x}) =$

A grid of binary digits (0s and 1s) arranged in a rectangular pattern, representing the machine language output of the C program. The binary digits are colored in shades of blue and green.

Magic word: ;

Magic word isomorphisms

(Σ_A, μ) **Markov chains** (Σ_B, ν)

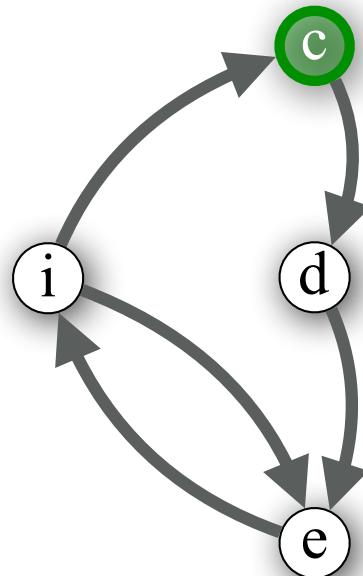
$$\phi: \Sigma_A \rightarrow \Sigma_B$$

Closed under compositions:

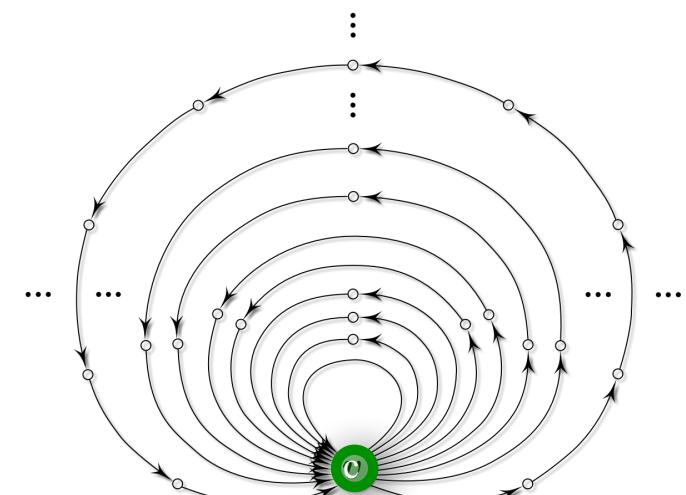


First return loop systems

- A adjacency matrix
- B results from A by removing row and column **c**



$\phi \rightarrow$



$$1 - f(z) = \frac{\det(I - zA)}{\det(I - zB)} = \frac{\zeta_B(z)}{\zeta_A(z)}$$

Magic word isomorphism problem

When are two Markov chains magic word isomorphic?

$$\Sigma_P \xrightarrow{MW} \Sigma_Q \Rightarrow (\Gamma_P, \Delta_P, c\Delta_P, \beta_P) = (\Gamma_Q, \Delta_Q, c\Delta_Q, \beta_Q)$$

$$\Sigma_P \xrightarrow{MW} \Sigma_Q \stackrel{?}{\Leftarrow} (\Gamma_P, \Delta_P, c\Delta_P, \beta_P) = (\Gamma_Q, \Delta_Q, c\Delta_Q, \beta_Q)$$

The greeks

$$\Gamma = \{\text{wt}(\gamma) : \gamma \text{ is a cycle}\}$$

$$\Delta = \left\{ \frac{\text{wt}(\gamma)}{\text{wt}(\gamma')} : \gamma, \gamma' \text{ are cycles of equal length} \right\}$$

$$c = \frac{\text{wt}(\gamma)}{\text{wt}(\gamma')} : \gamma, \gamma' \text{ are cycles and } \ell(\gamma) = \ell(\gamma') + 1$$

$$\beta(t) = \text{Spectral radius}(P_t)$$

Magic word isomorphism problem

When are two Markov chains magic word isomorphic?

$$\Sigma_P \xrightarrow{MW} \Sigma_Q \Rightarrow (\Gamma_P, \Delta_P, c\Delta_P, \beta_P) = (\Gamma_Q, \Delta_Q, c\Delta_Q, \beta_Q)$$

$$\Sigma_P \xrightarrow{MW} \Sigma_Q \stackrel{?}{\Leftarrow} (\Gamma_P, \Delta_P, c\Delta_P, \beta_P) = (\Gamma_Q, \Delta_Q, c\Delta_Q, \beta_Q)$$

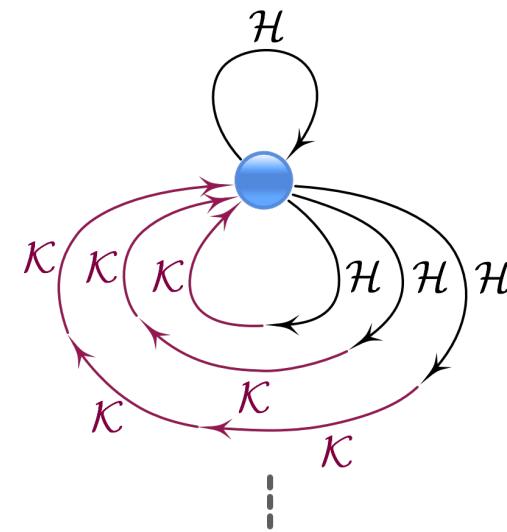
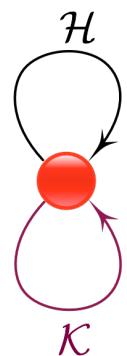


$$\Sigma_A \xrightarrow{MW} \Sigma_B \stackrel{\checkmark}{\Leftarrow} h(A) = h(B)$$

Basic move

$$\mathcal{F} = \mathcal{H} + \mathcal{K}$$

$$\mathcal{G} = \mathcal{H} \times \text{SEQ}(\mathcal{K})$$



$$F(z) = H(z) + K(z)$$

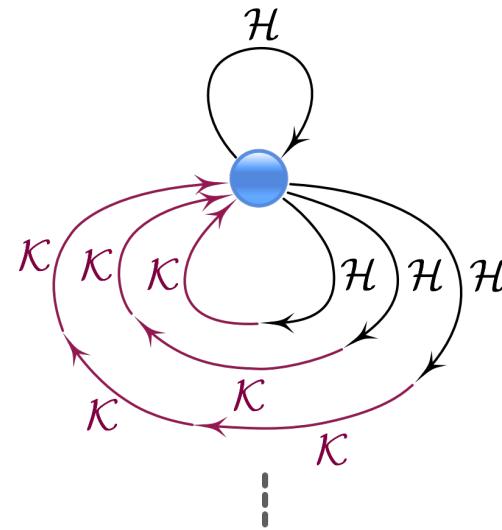
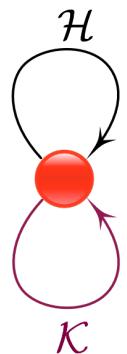
$$G(z) = H(z)(1 + K(z) + K(z)^2 + \dots)$$

$$h(\mathcal{K}) < h(\mathcal{F}) \quad \Rightarrow \quad \mathcal{F} \stackrel{MW}{\approx} \mathcal{G}$$

Basic move

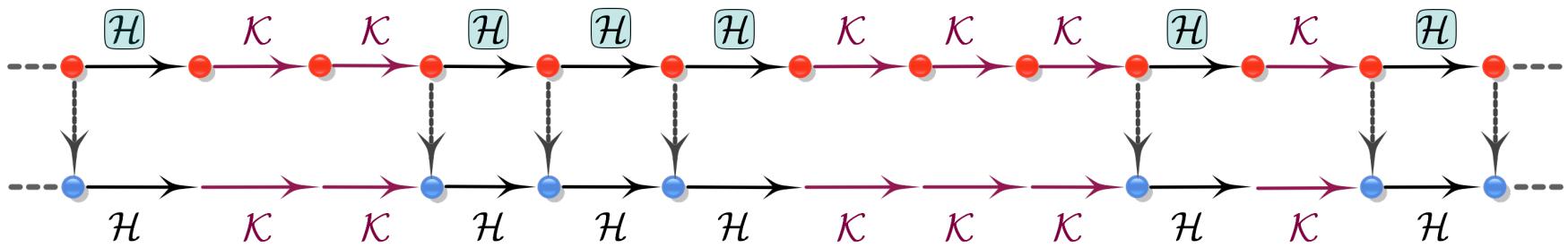
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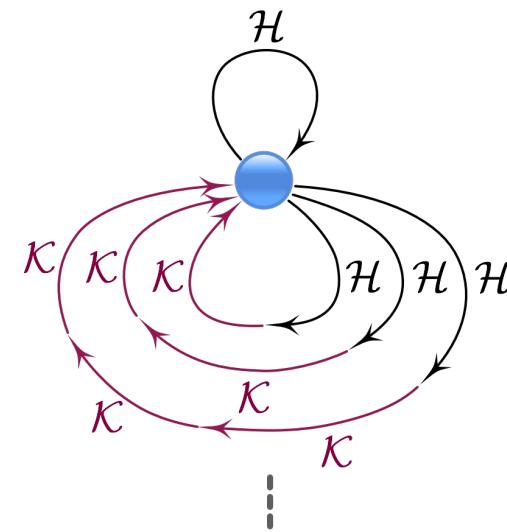
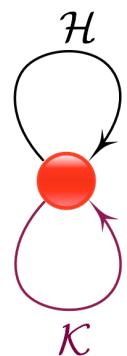
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Basic move

$$\mathcal{F} = \mathcal{H} + \mathcal{K}$$

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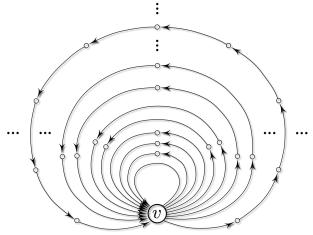
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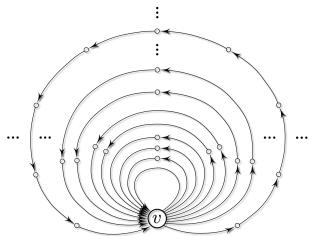
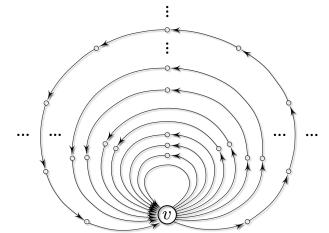
$$\mathcal{K} = \mathcal{Z}^n \quad \Rightarrow$$

- $G_k = F_k \quad \forall k < n$
- $G_n = F_n - 1$

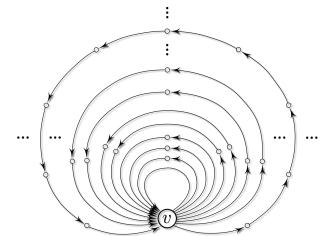
Loops blowing algorithm



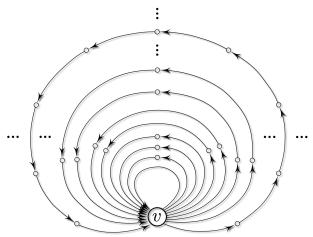
$$f^{\langle 0 \rangle}(z) = \sum_{n=1}^{\infty} f_n^{\langle 0 \rangle} z^n$$



$$f^{\langle 1 \rangle}(z) = \sum_{n=1}^{\infty} f_n^{\langle 1 \rangle} z^n$$



⋮



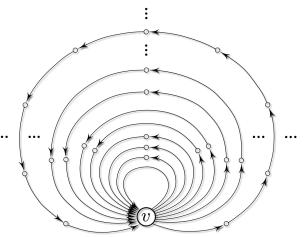
$$f^{\langle \infty \rangle}(z) = \sum_{n=1}^{\infty} f_n^{\langle \infty \rangle} z^n$$

$$g^{\langle 0 \rangle}(z) = \sum_{n=1}^{\infty} g_n^{\langle 0 \rangle} z^n$$

$$g^{\langle 1 \rangle}(z) = \sum_{n=1}^{\infty} g_n^{\langle 1 \rangle} z^n$$

⋮

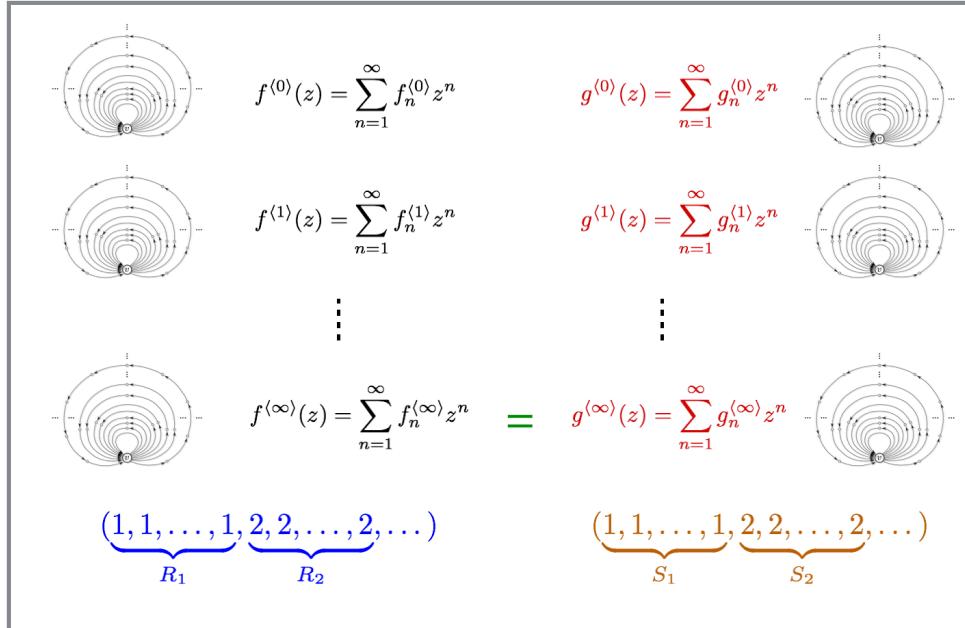
$$g^{\langle \infty \rangle}(z) = \sum_{n=1}^{\infty} g_n^{\langle \infty \rangle} z^n$$



$$(\underbrace{1, 1, \dots, 1}_{R_1}, \underbrace{2, 2, \dots, 2}_{R_2}, \dots)$$

$$(\underbrace{1, 1, \dots, 1}_{S_1}, \underbrace{2, 2, \dots, 2}_{S_2}, \dots)$$

Magic word isomorphism



- $\limsup R_n^{1/n} < \lambda$
- $R_n \leq f_n^{(0)} \quad \forall n \geq 1$
- $R_1 < f_1^{(0)}$
- $\limsup S_n^{1/n} < \lambda$
- $S_n \leq g_n^{(0)} \quad \forall n \geq 1$
- $S_1 < g_1^{(0)}$

Magic word isomorphism

Given $1 \leq \beta < \lambda$ there exist $N \gg 0$ and $F, G \in \mathbb{Z}_+[[z]]$ such that



- $\Sigma_f \stackrel{MW}{\approx} \Sigma_F$ and $\Sigma_g \stackrel{MW}{\approx} \Sigma_G$
- $|\mathcal{O}_n(\Sigma_F)| = |\mathcal{O}_n(\Sigma_G)| = 0 \quad \forall n < N$
- $|\mathcal{O}_n(\Sigma_f)| - |\mathcal{O}_n(\Sigma_g)| = |\mathcal{O}_n(\Sigma_F)| - |\mathcal{O}_n(\Sigma_G)| \quad \forall n \geq N$
- $\min\{F_n, G_n\} \geq \beta^n \quad \forall n \geq N$

- | | |
|--|--|
| <ul style="list-style-type: none">• $\limsup R_n^{1/n} < \lambda$• $R_n \leq f_n^{\langle 0 \rangle} \quad \forall n \geq 1$• $R_1 < f_1^{\langle 0 \rangle}$ | <ul style="list-style-type: none">• $\limsup S_n^{1/n} < \lambda$• $S_n \leq g_n^{\langle 0 \rangle} \quad \forall n \geq 1$• $S_1 < g_1^{\langle 0 \rangle}$ |
|--|--|

Magic word isomorphism

Exponencial Recurrence

Transient:

$$f(1/\lambda) < 1$$

Recurrent:

$$f(1/\lambda) = 1$$

Positive Recurrent:

$$f(1/\lambda) = 1 \quad \& \quad \sum_{n=1}^{\infty} n f_n / \lambda^n < \infty$$

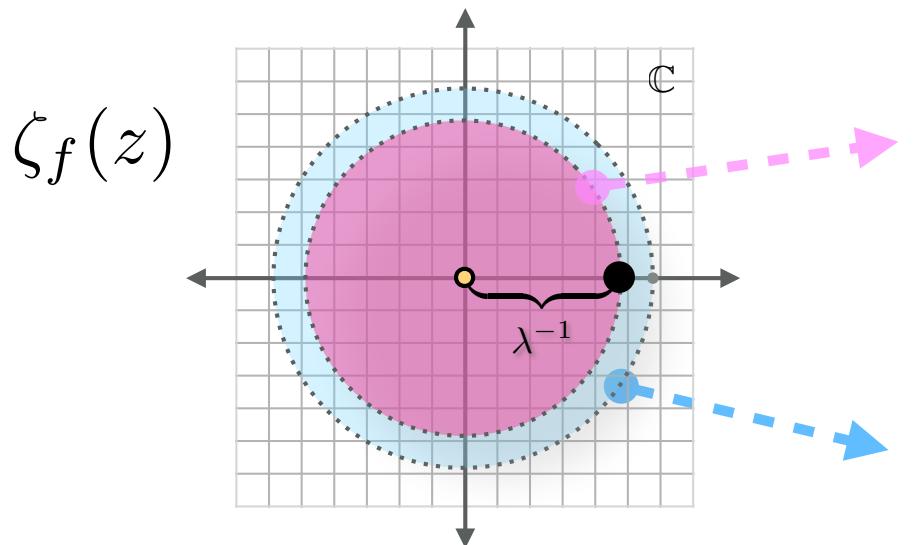
Exponencial Recurrence:

$$\limsup_{n \rightarrow \infty} f_n^{1/n} < \lambda$$

- $\limsup R_n^{1/n} < \lambda$
- $R_n \leq f_n^{(0)} \quad \forall n \geq 1$
- $R_1 < f_1^{(0)}$
- $\limsup S_n^{1/n} < \lambda$
- $S_n \leq g_n^{(0)} \quad \forall n \geq 1$
- $S_1 < g_1^{(0)}$

Magic word isomorphism

Exponencial Recurrence



$$f(z) = \sum_{n=1}^{\infty} f_n z^n$$

$$\zeta_f(z) = \frac{1}{1 - f(z)}$$

- $\limsup R_n^{1/n} < \lambda$
- $R_n \leq f_n^{(0)} \quad \forall n \geq 1$
- $R_1 < f_1^{(0)}$
- $\limsup S_n^{1/n} < \lambda$
- $S_n \leq g_n^{(0)} \quad \forall n \geq 1$
- $S_1 < g_1^{(0)}$

Magic word isomorphism

Exponencial Recurrence

$$\limsup \left| |\mathcal{O}_n(\Sigma_f)| - |\mathcal{O}_n(\Sigma_g)| \right|^{1/n} < \lambda$$

- $\limsup R_n^{1/n} < \lambda$
- $R_n \leq f_n^{\langle 0 \rangle} \quad \forall n \geq 1$
- $R_1 < f_1^{\langle 0 \rangle}$
- $\limsup S_n^{1/n} < \lambda$
- $S_n \leq g_n^{\langle 0 \rangle} \quad \forall n \geq 1$
- $S_1 < g_1^{\langle 0 \rangle}$

Magic word isomorphism

Exponencial Recurrence

$$\gamma = \limsup \left| |\mathcal{O}_n(\Sigma_f)| - |\mathcal{O}_n(\Sigma_g)| \right|^{1/n} < \lambda$$
$$\gamma < \beta < \lambda$$

- | | |
|--|--|
| <ul style="list-style-type: none">• $\limsup R_n^{1/n} < \lambda$• $R_n \leq f_n^{\langle 0 \rangle} \quad \forall n \geq 1$• $R_1 < f_1^{\langle 0 \rangle}$ | <ul style="list-style-type: none">• $\limsup S_n^{1/n} < \lambda$• $S_n \leq g_n^{\langle 0 \rangle} \quad \forall n \geq 1$• $S_1 < g_1^{\langle 0 \rangle}$ |
|--|--|

Magic word isomorphism

Exponencial Recurrence

$$\gamma = \limsup \left| |\mathcal{O}_n(\Sigma_f)| - |\mathcal{O}_n(\Sigma_g)| \right|^{1/n} < \lambda$$

$$N \gg 0 \quad \Rightarrow \quad \left| |\mathcal{O}_n(\Sigma_F)| - |\mathcal{O}_n(\Sigma_G)| \right|^{1/n} \leq \beta^n - 1$$

- $\limsup R_n^{1/n} < \lambda$
- $R_n \leq f_n^{\langle 0 \rangle} \quad \forall n \geq 1$
- $R_1 < f_1^{\langle 0 \rangle}$
- $\limsup S_n^{1/n} < \lambda$
- $S_n \leq g_n^{\langle 0 \rangle} \quad \forall n \geq 1$
- $S_1 < g_1^{\langle 0 \rangle}$

Magic word isomorphism

Exponencial Recurrence

$$R_n = \max\{0, |\mathcal{O}_n(\Sigma_F)| - |\mathcal{O}_n(\Sigma_G)|\}$$

$$S_n = \max\{0, |\mathcal{O}_n(\Sigma_G)| - |\mathcal{O}_n(\Sigma_F)|\}$$



- $\limsup R_n^{1/n} < \lambda$
- $R_n \leq f_n^{\langle 0 \rangle} \quad \forall n \geq 1$
- $R_1 < f_1^{\langle 0 \rangle}$
- $\limsup S_n^{1/n} < \lambda$
- $S_n \leq g_n^{\langle 0 \rangle} \quad \forall n \geq 1$
- $S_1 < g_1^{\langle 0 \rangle}$

Magic word isomorphism

Exponencial Recurrence

$$R_n = \max\{0, |\mathcal{O}_n(\Sigma_F)| - |\mathcal{O}_n(\Sigma_G)|\}$$

$$S_n = \max\{0, |\mathcal{O}_n(\Sigma_G)| - |\mathcal{O}_n(\Sigma_F)|\}$$



MR2222728 (2006m:37011) Reviewed

Boyle, Mike(1-MD); Buzzi, Jerome(F-POLY-CMT); Gómez,
Ricardo(MEX-NAM-IN)

Almost isomorphism for countable state Markov shifts.
(English summary)

J. Reine Angew. Math. 592 (2006), 23–47.

Magic word isomorphism

Exponencial Recurrence

$$R_n = \max\{0, |\mathcal{O}_n(\Sigma_F)| - |\mathcal{O}_n(\Sigma_G)|\}$$

$$S_n = \max\{0, |\mathcal{O}_n(\Sigma_G)| - |\mathcal{O}_n(\Sigma_F)|\}$$



MR3271228 Reviewed

Boyle, Mike(1-MD); Buzzi, Jérôme(F-PARIS11-M); Gómez, Ricardo(MEX-NAM-IM)

Borel isomorphism of SPR Markov shifts. (English summary)

Colloq. Math. 137 (2014), no. 1, 127–136.

37A35 (03E15 37B10)

Magic word isomorphism

Exponencial Recurrence

$$R_n = \max\{0, |\mathcal{O}_n(\Sigma_F)| - |\mathcal{O}_n(\Sigma_G)|\}$$

$$S_n = \max\{0, |\mathcal{O}_n(\Sigma_G)| - |\mathcal{O}_n(\Sigma_F)|\}$$



MR0482707 (58 #2763) Reviewed

Parry, William

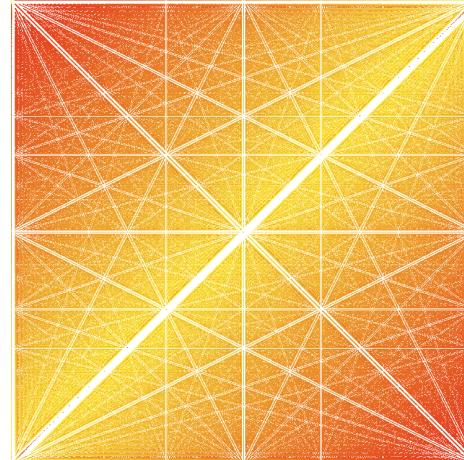
A finitary classification of topological Markov chains and sofic systems.

Bull. London Math. Soc. **9** (1977), no. 1, 86–92.

54H20 (28A65 58F15)

Magic word isomorphism

Markov chains



Exponential recurrence

Magic word isomorphism

Markov chains



Exponential recurrence

Other probabilistic regimes:

NR and TR

Shift Dominant Equivalence

$$\Sigma_f \stackrel{SDE}{\approx} \Sigma_g$$

$\exists K \geq 0$ such that $\forall n \geq 0$

$$f_n \leq g_{n+K} \quad \text{and} \quad g_n \leq f_{n+K}$$

$h < \infty$	ER $h =$ SDE = MW
	PR $h =$ SDE \ll MW
	N $h \ll$ SDE \Leftarrow MW
	T $h \ll$ SDE \Leftarrow MW
$h = \infty$	LZF $h \ll$ SDE \Leftarrow AI

Shift Dominant Equivalence

What is the asymptotic growth of
sequence schemas of polylogarithm functions?



$h < \infty$	ER	$h =$	SDE	$=$	MW
	PR	$h =$	SDE	\ll	MW
	N	$h \ll$	SDE	\Leftarrow	MW
	T	$h \ll$	SDE	\Leftarrow	MW
$h = \infty$		LZF	$h \ll$	SDE	\Leftarrow
					AI

Shift Dominant Equivalence

What is the asymptotic growth of
sequence schemas of polylogarithm functions?

$$\text{Li}_\alpha(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\alpha}$$

$$F(z) := \frac{\text{Li}_{\alpha,r}(z)}{\text{Li}_{\alpha,r}(1)} \quad \text{and} \quad S(z) := \frac{1}{1 - F(z)}.$$

Theorem [ **Ward**, G. Monday]

$$[z^n]S(z) \sim \frac{(1 - \alpha) \sin(\pi\alpha) \text{Li}_\alpha(1) n^{\alpha-2}}{\pi}$$

Shift Dominant Equivalence

Catalog of asymptotic analysis of subcritical sequence schemas???

$h < \infty$	ER	$h =$	SDE	$=$	MW
	PR	$h =$	SDE	\ll	MW
	N	$h \ll$	SDE	\leq	MW
	T	$h \ll$	SDE	\leq	MW
$h = \infty$	LZF	$h \ll$	SDE	\leq	AI

???

Theorem [ Ward, G. Monday]

$$[z^n]S(z) \sim \frac{(1 - \alpha) \sin(\pi\alpha) \text{Li}_\alpha(1) n^{\alpha-2}}{\pi}$$

Shift Dominant Equivalence

MR1678788 (2000a:05015) Reviewed

Flajolet, Philippe(F-INRIA)

Singularity analysis and asymptotics of Bernoulli sums.

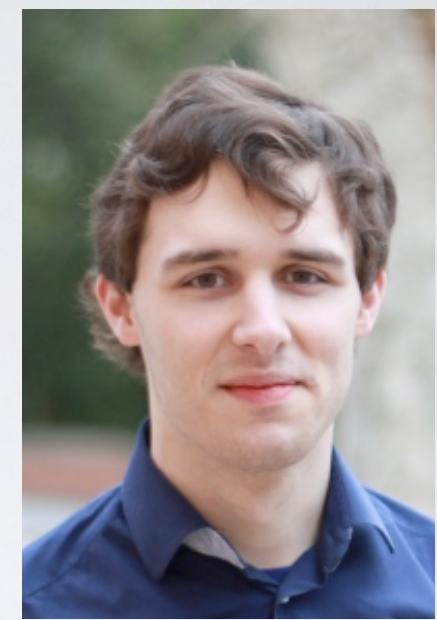
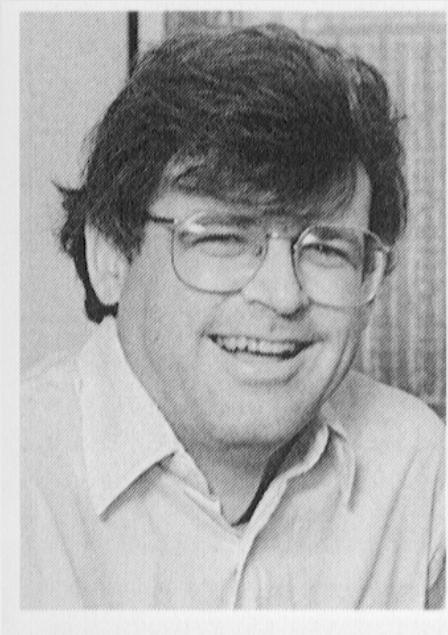
(English summary)

Theoret. Comput. Sci. 215 (1999), no. 1-2, 371–381.

05A16 (05A15 68Q25)

Theorem [ **Ward**, G. Monday]

$$[z^n]S(z) \sim \frac{(1 - \alpha) \sin(\pi\alpha) \text{Li}_\alpha(1) n^{\alpha-2}}{\pi}$$



THANK YOU !!!

