Pattern Distributions in Random Restricted Permutations

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June 19, 2017

Plotting Permutations

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Definition

If π is a permutation of length n, then the **plot** of π is the set of points

$$\{(1, \pi(1)), (2, \pi(2)), \cdots (n, \pi(n))\} \subset \mathbb{R}^2$$

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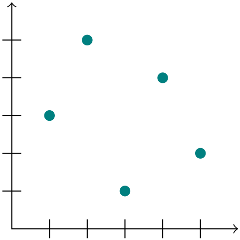

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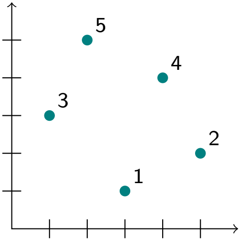

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Dots on a Plane

Definition

Let A and B be two sets of n points in $\mathbb{R}^2$, each with the property that no two points lie on the same horizontal or vertical line.
Say that A is **order isomorphic** to B (denoted $A \sim B$) if A can be transformed into B by stretching, contracting, and translating the axes horizontally and vertically.

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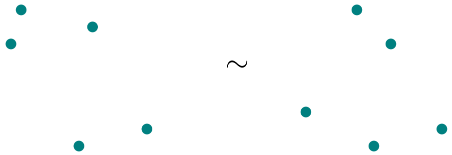

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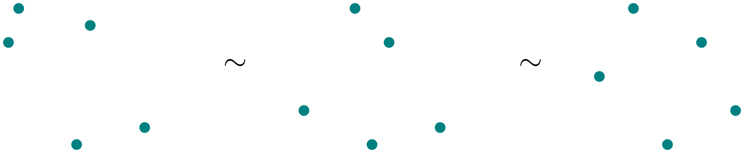

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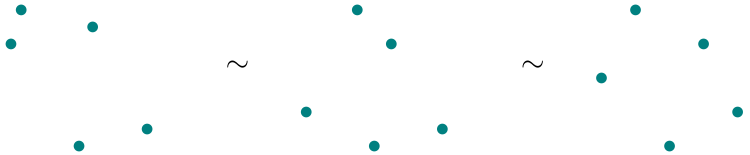

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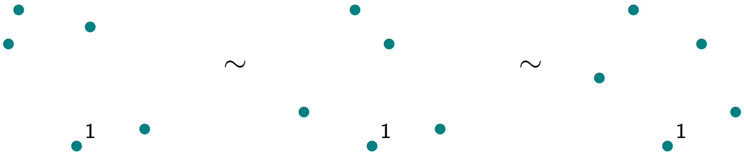

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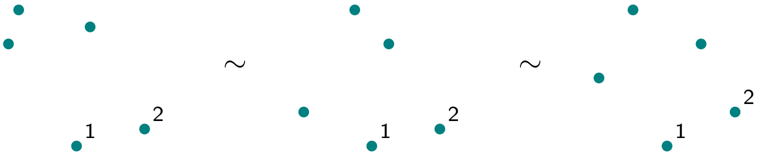

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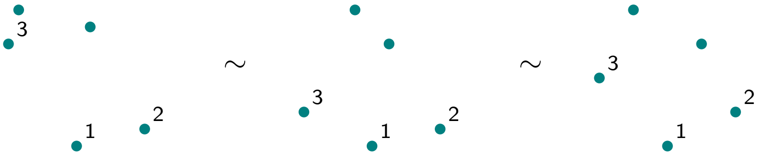

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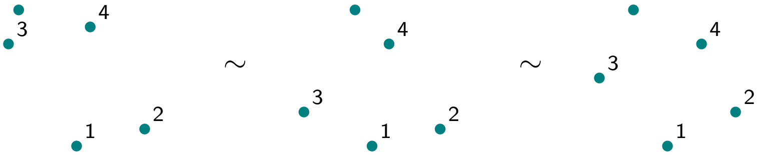

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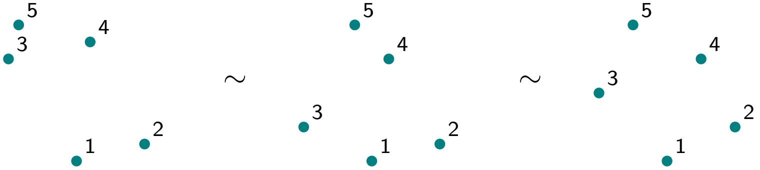

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Permutation Classes

Definition

A permutation π contains σ **as a pattern** if the plot of σ is order isomorphic to a subset of the plot of π.

Definition

The set of all permutations ordered by pattern containment is known as the **permutation pattern poset**. A downset of this poset is called a **permutation class**.

Definition

The set of permutations **avoiding** a specified pattern (or set of patterns) σ is denoted $\mathsf{Av}(\sigma)$.

Av(132) and Av(123)

Question

What do 132- and 123-avoiding permutations look like?

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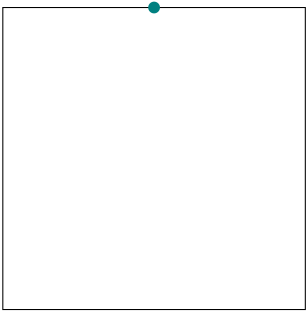

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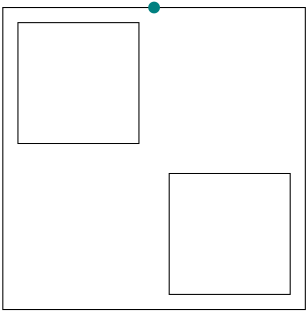

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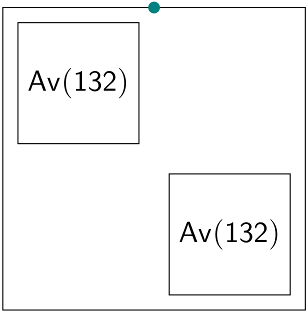

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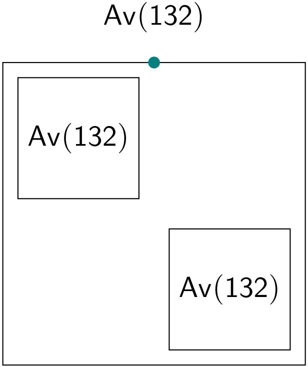

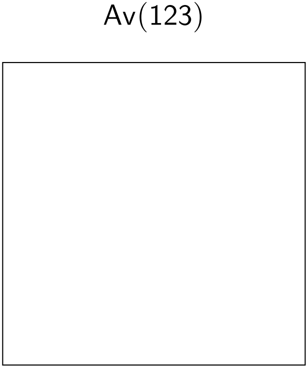

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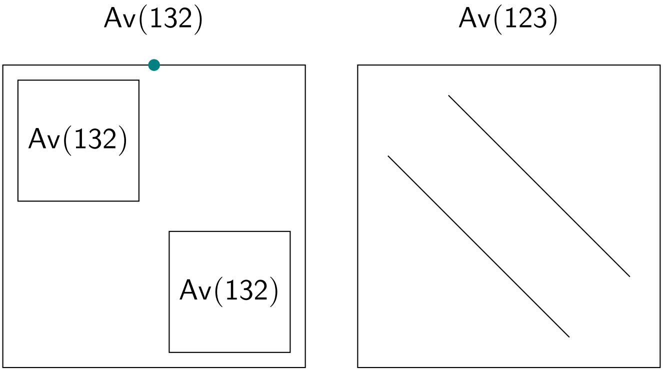

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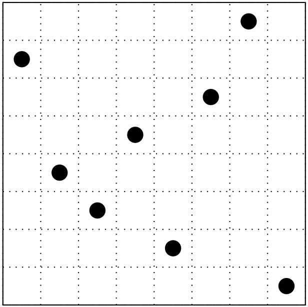

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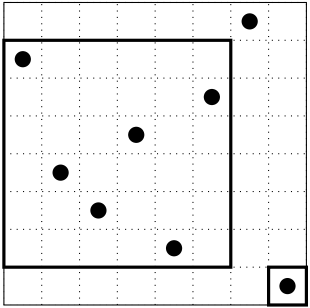

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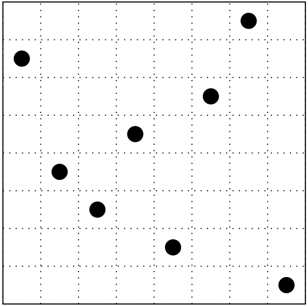

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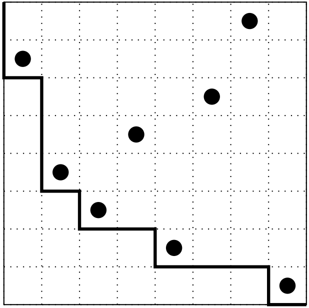

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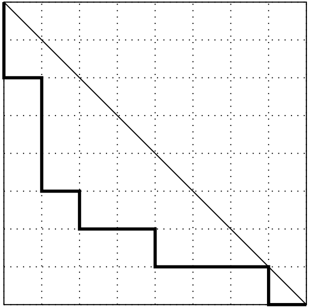

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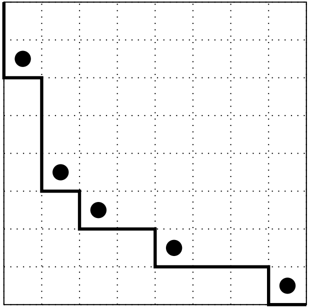

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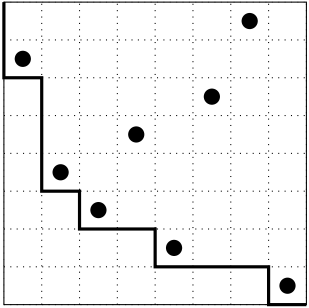

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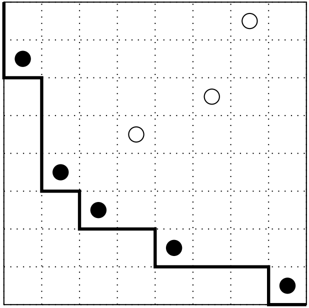

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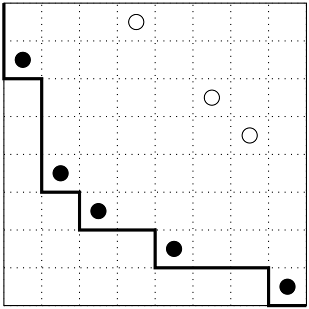

$\mathrm{Av}(132) \mapsto \mathrm{Av}(123)$

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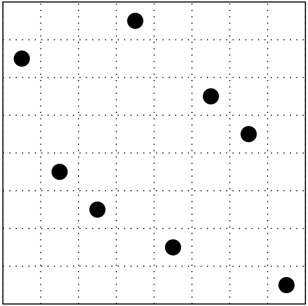

Av(123)

Theorem

$Av(132)$ is in bijection with Av(123).

Pattern Occurrences

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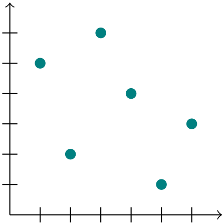

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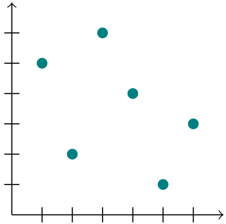

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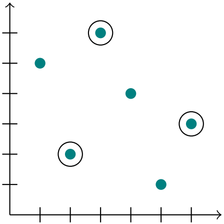

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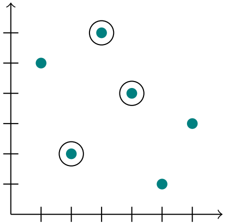

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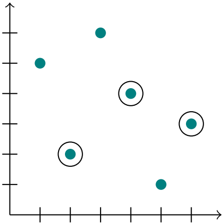

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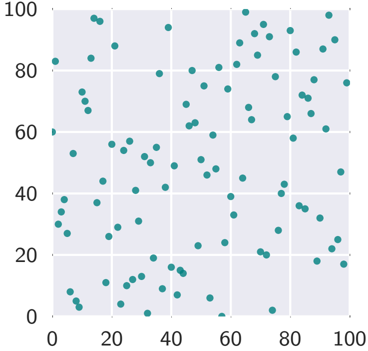
Theorem (Janson, Nakamura, Zeilberger 2013)

For a randomly selected permutation of length n and two patterns σ and ρ, the random variables ν_σ and ν_ρ are asymptotically jointly normally distributed as $n \to \infty$.

Random Permutations



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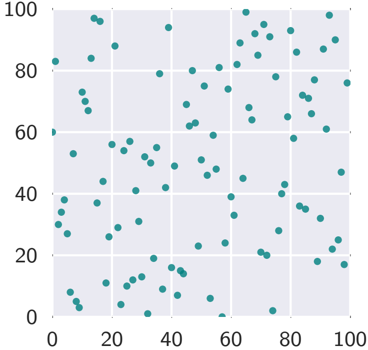

ν_{12}	ν_{21}	Avg
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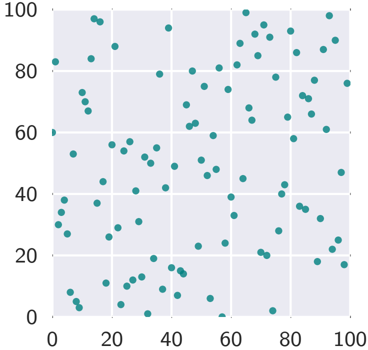

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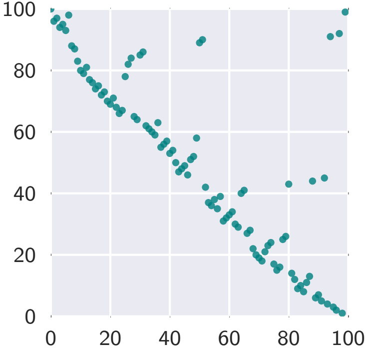
ν_{123}	ν_{132}	ν_{213}	ν_{231}	ν_{312}	ν_{321}	Avg
35357	30063	31414	22321	23348	19197	26950

Random Restricted Permutations



ν_{12}	ν_{21}	Avg
685	4265	2475

ν_{123}	ν_{132}	ν_{213}	ν_{231}	ν_{312}	ν_{321}	Avg
2426	0	14874	15208	14896	114296	26950

Exact Behavior

Fact

In $\mathfrak{S}_n$, the number of occurrences of a specific pattern depends only on the length of the pattern. That is, for a pattern $\sigma \in \mathfrak{S}_k$, we have

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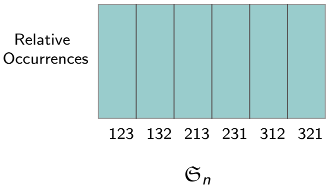

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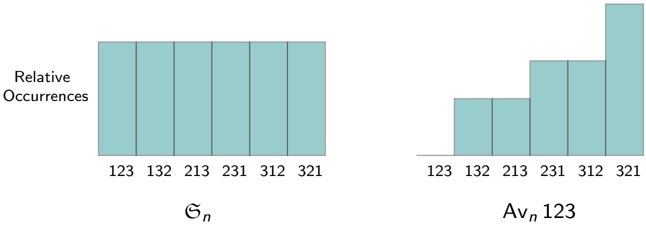

Equipopularity

Definition

The **popularity** of a pattern σ in a class C is the sequence

$$\nu_\sigma(C_1),\ \nu_\sigma(C_2),\ \nu_\sigma(C_3),\ \nu_\sigma(C_4),\ \ldots.$$

Definition

Patterns are said to be **equipopular** if they have the same number of occurrences (within a specified set or across two different sets).

Equipopularity — Warm up

Fact

For a class C and a pattern σ, we have

$$\nu_\sigma(C_n) = |\{(\pi;\sigma^*) : \pi \in C_n,\ \sigma^* \prec \pi\}|.$$

Equipopularity — Warm up

Fact

For a class C and a pattern σ, we have

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Proposition

In the class $\mathrm{Av}(132)$, σ and σ^{-1} are equipopular.

Proof.

$$(\pi,\sigma) \mapsto (\pi^{-1},\sigma^{-1}).$$

□

History

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Theorem (Bóna 2010)

Within the class $\mathrm{Av}(132)$:

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Theorem (Chua, Sankar 2013)

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History (in Pictures)

Separable Permutations

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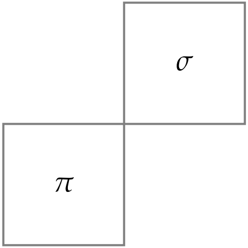
Alternate Definition

The separable permutations are those which can be constructed via arbitrary **skew** and **direct sums** of the permutation 1.

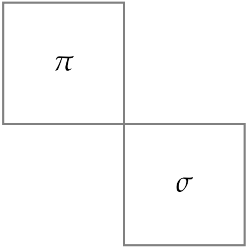
Separable Permutations

Definition

Given two permutations π and σ, their **direct sum** ($\pi \oplus \sigma$) and **skew sum** ($\pi \ominus \sigma$) are defined as follows:



$\pi \oplus \sigma$



$\pi \ominus \sigma$

Separable Permutations

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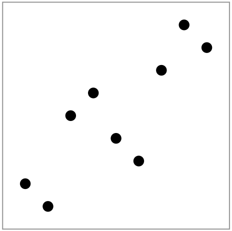

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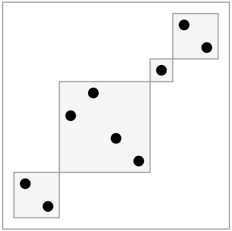

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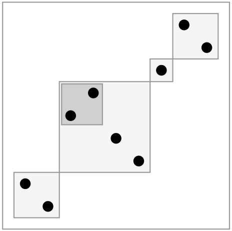
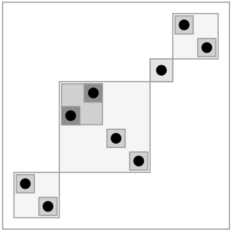


Separable Permutations

$$\pi = 215643798 = \Big(1 \ominus 1\Big) \oplus \Big((1 \oplus 1) \ominus 1 \ominus 1\Big) \oplus 1 \oplus \Big(1 \ominus 1\Big).$$



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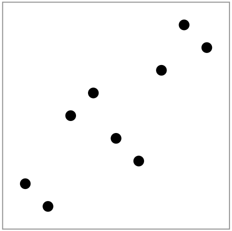

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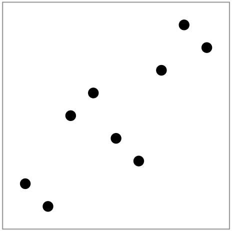

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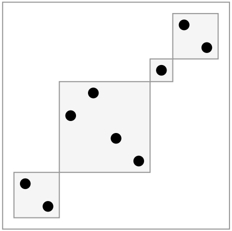

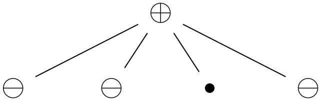

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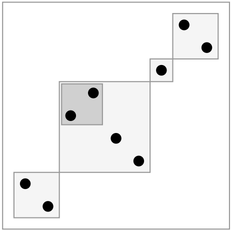

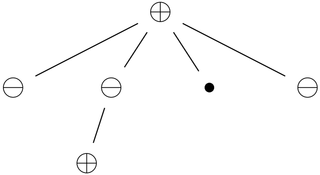

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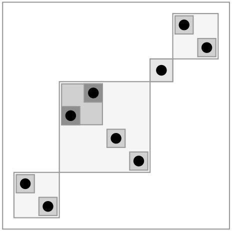
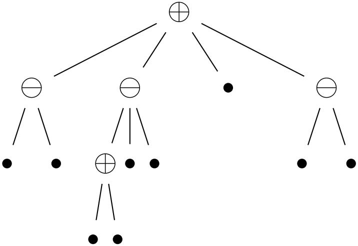


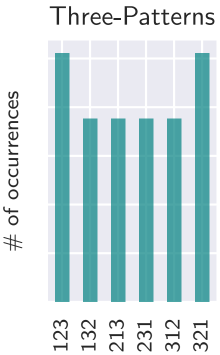


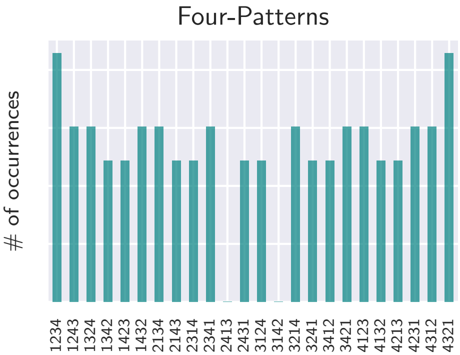
Popularity of Separable Permutations — A Classification Theorem

Theorem (Albert, H, Pantone)

Two patterns are equipopular in the separables if and only if they **have the same structure**. Further, the equipopularity classes are in bijection with the set of integer partitions.

Popularity in the Separables





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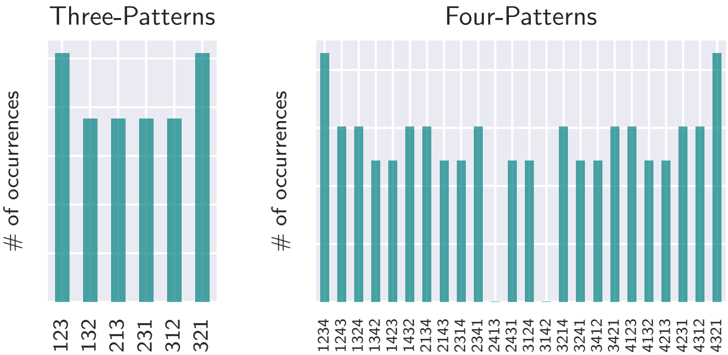

Corollary

The number of distinct levels in these histograms is equal to the number of integer partitions.

Tree Patterns

Proof Idea

Equipopularity can be characterized by tree structure.

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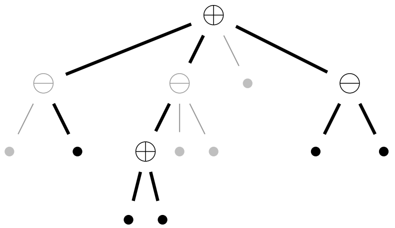
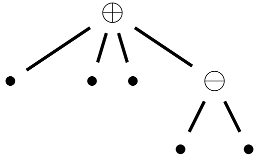
Pattern Containment:





Strategy

Part 1

Find the operations on trees which preserve popularity.

Part 2

Show that equipopularity implies that their trees are related by one of these operations.

Preserving Popularity — Tree Operations

Lemma

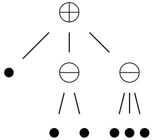
The following operations preserve popularity:

- Swapping $\oplus$ and $\ominus$ signs
- Rearranging the children of any node
- Tree rotation

Sketch of Proof

Preserving Popularity — Tree Operations

Sketch of Proof



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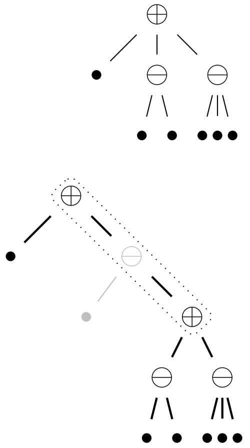
Sketch of Proof



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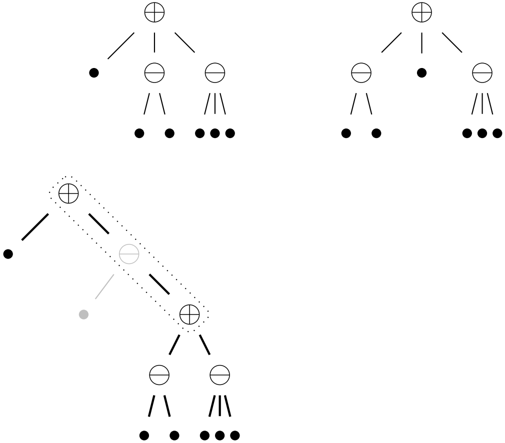

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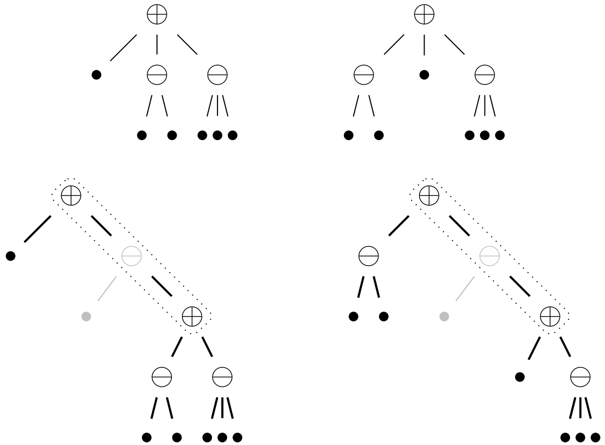
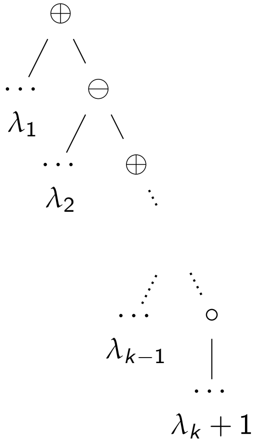


Canonical Representatives



$$\lambda := \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$$

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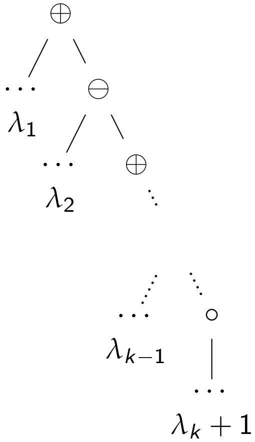

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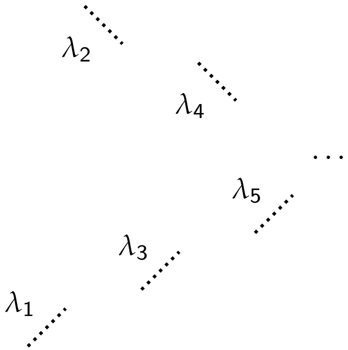

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The Other Direction

Lemma

Different partitions lead to different enumerations.

Rough Sketch of Proof

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Given any arbitrary pattern, we can factor its popularity generating function into the popularity generating functions for monotone runs.

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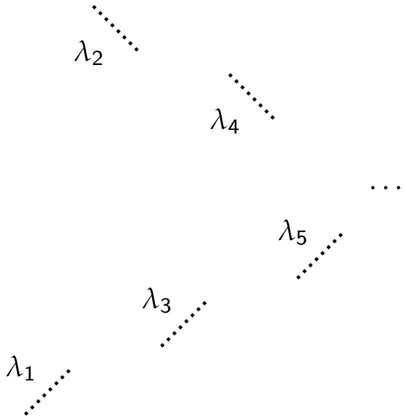

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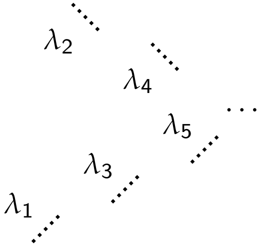
- Notice (or let Sage/Maple/Mathematica/Singular tell you) that these are related to the **Gegenbauer polynomials**, a family of orthogonal polynomials.
- Use the orthogonality of these polynomials to uniquely factor any product.

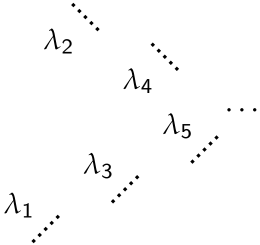
Corollary



$$\leftrightarrow \; \lambda_1 + \lambda_2 + \cdots + \lambda_{k-1} + (\lambda_k - 1)$$

Length n canonical representative $\leftrightarrow$ partition of the integer $n-1$.

Corollary



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(We also have a process for building generating functions and calculating asymptotics for each of these popularities)

Open Questions/Directions

- Are there other instances of non-trivial equipopularity within other permutation classes?
- Are there other instances of equipopularity across **different** permutation classes?
- Are there non-trivial examples of **equidistribution** of pattern occurrences within or across permutation classes?
- Miner and Pak recently introduced the notion of the asymptotic **shape** of permutation classes, and calculated some examples. How can we similarly characterize arbitrary classes?
- Bassino, Bouvel, Féray, Gerin, and Pierrot recently calculated the distribution of separable patterns by expressing the class as a "Brownian separable permuton." Is this possible for other classes?

Thank You!