

DePoissonization: “Vingt Ans Apres” Closing the Loop

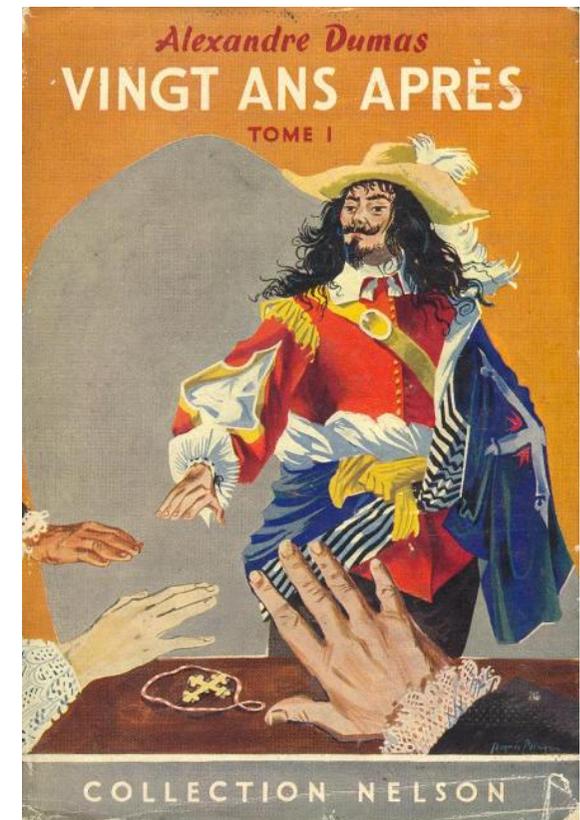
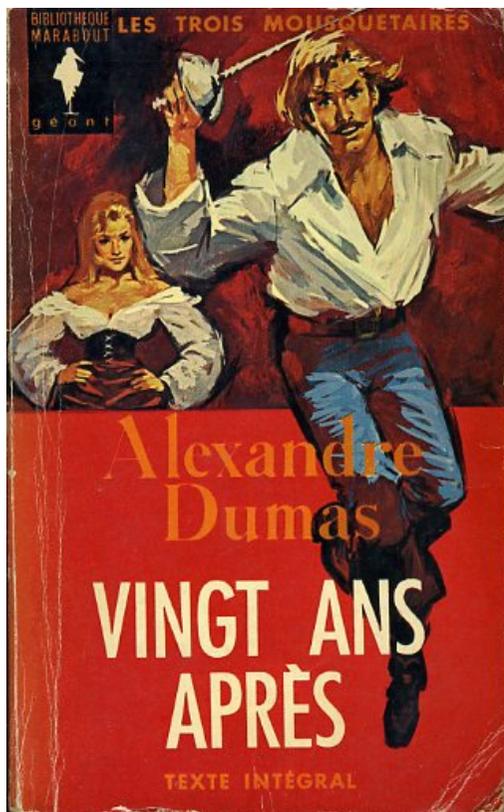
Philippe Jacquet
Nokia Bell-Labs

Joint or parallel works with Szpankowski,
Regnier, Huang, Fuchs, and many other.

Dedicated to Philippe Flajolet

Not so Long Time ago (please)

P. Jacquet, W. Szpankowski, Analytical dePoissonization and its application, TCS, 1998



The theorem

- Under \mathcal{QS} conditions the Poisson generating function

satisfies
$$f(z) = \sum_n a_n \frac{z^n}{n!} e^{-z}$$

$$a_n = f(n) + O(f(n)/n)$$

Also
$$a_n = \sum_{i=0}^k \sum_{j=1}^{k+i} b_{ij} n^i f^{(j)}(n) + O(n^{-k} f(n))$$

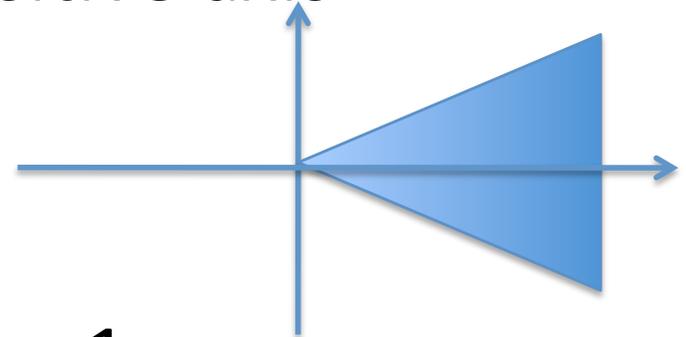
with
$$\sum_i \sum_j b_{ij} x^i y^j = \exp(x \log(1+y) - xy)$$

Related to Poisson-Charlier and Laguerre polynomials

\mathcal{QS} condition

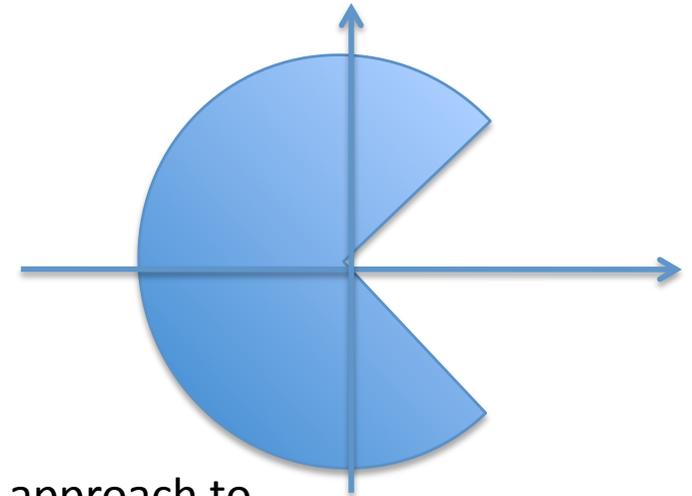
- (I) In a cone around the real positive axis

$$f(z) = O(|z|^\beta), \quad |z| \rightarrow \infty$$



- (O) Outside the cone for some $\alpha < 1$

$$f(z)e^z = O(e^{\alpha|z|}), \quad |z| \rightarrow \infty$$



– What if (O) not satisfied?

- eg $a_n = (-1)^n, f(z) = e^{-2z}$

Fuchs, M., Hwang, H. K., & Zacharovas, V. (2014). An analytic approach to the asymptotic variance of trie statistics and related structures.

Theoretical Computer Science, 527, 1-36.

Extended depoissonization

- Exponential dePo

- Trie size normal limiting law $a_n = E[u^{S_n}]$, when $|u| \geq 1$

- Diagonal dePo $a_{n,n} = E[u_n^{S_n}]$

- Polynomial cone $f(z) = O(|z|^\beta)$, when $|\arg(z)| \leq C|z|^\delta$

- DST path length limiting law

- LZ compression redundancy rate

- Double, multiple, dePo

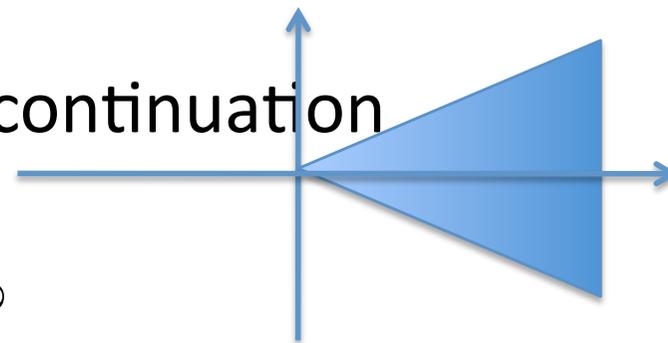
- Joint Complexity $f(z_1, z_2) = \sum_{n_1} \sum_{n_2} a_{n_1, n_2} \frac{z_1}{n_1!} \frac{z_2}{n_2!} e^{-z_1 - z_2}$

The Closing the Loop \mathcal{CL} theorem

- The \mathcal{JS} condition is strictly equivalent to \mathcal{CL} condition

- The sequence a_n has an analytic continuation $a(x)$ such that

$$a(x) = O(|x|^\beta), \quad |x| \rightarrow \infty$$



in a cone around the real axis (not necessarily the same as for \mathcal{JS} condition) and

$$f(z) = \sum_{i=0}^k \sum_{j=1}^{k+i} c_{ij} z^i a^{(j)}(z) + O(z^{\beta-k-1})$$

With (related to Stirling number 2nd kind) $\sum_i \sum_j c_{ij} x^i y^j = \exp(x(e^y - 1) - xy)$

Consequence

- If $f(z) = \sum_n a_n \frac{z^n}{n!} e^{-z}$ satisfies \mathcal{QS} then Hadamard product $f * f(z) = \sum_n a_n^2 \frac{z^n}{n!} e^{-z}$ satisfies \mathcal{QS}

- $f^{*k}(z) = \sum_n a_n^k \frac{z^n}{n!} e^{-z}$ too

- $\sum_n \log a_n \frac{z^n}{n!} e^{-z}$ too

Fuchs, M., Hwang, H. K., & Zacharovas, V. (2014). An analytic approach to the asymptotic variance of trie statistics and related structures.

Theoretical Computer Science, 527, 1-36.

Exemple: entropy computation

$$h_n = - \sum_{k=0}^{k=n} p^k (1-p)^{n-k} \binom{n}{k} \log \left\{ p^k (1-p)^{n-k} \binom{n}{k} \right\}$$

$$H(z) = pz \log p + qz \log(1-p) + f(pz) + f((1-p)z)$$

$$f(z) = \sum_{n \geq 0} \log n! \frac{z^n}{n!} e^{-z}$$

Jacquet, P., & Szpankowski, W. (1999). Entropy computations via analytic depoissonization. *IEEE Transactions on Information Theory*, 45(4), 1072-1081.

Proof $\mathcal{S} \Rightarrow \mathcal{CL}$

- In fact the easiest way of the proof

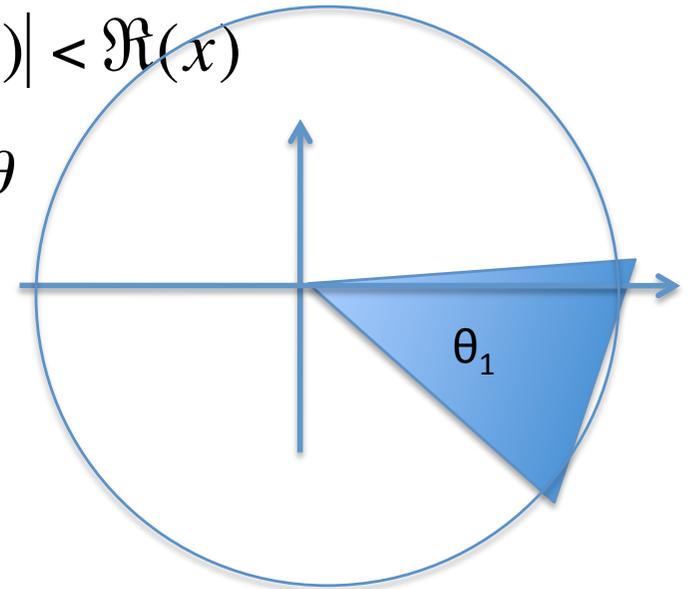
$$a_n = \frac{n!}{2i\pi} \oint f(z) e^z z^{-n-1} dz$$

$$a(x) = \frac{\Gamma(1+x)}{2\pi} x^{-x} \int_{-\pi}^{\pi} f(xe^{i\theta}) \exp((e^{i\theta} - i\theta)x) d\theta$$

Proof $\mathcal{IS} \Rightarrow \mathcal{CL}$

- Select cone such that $\alpha|x| + \pi|\Im(x)| < \Re(x)$

$$\begin{aligned}
 a(x) &= \frac{\Gamma(1+x)}{e^{-x}} x^{-x} O\left(e^{\int_0^{\pi} \theta(xe^{i\theta}) \exp(i\theta) d\theta}\right) \\
 &+ \frac{\Gamma(1+x)}{2\pi} \int_{|\theta + \arg(x)| > \theta_1} \theta(xe^{i\theta}) \exp(i\theta) d\theta \\
 &+ \frac{\Gamma(1+x)}{2\pi} \int_{|\theta + \arg(x)| \leq \theta_1} \theta(xe^{i\theta}) \exp(i\theta) d\theta
 \end{aligned}$$



Jacquet, P. (2014). Trie structure for graph sequences.

Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms, 181.

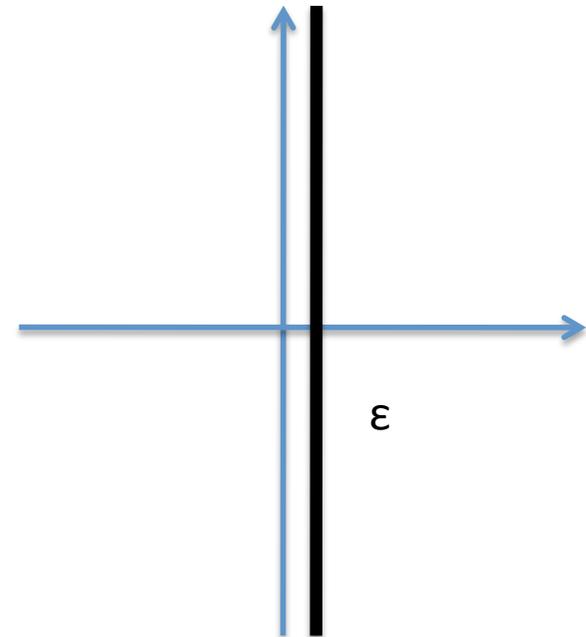
Proof $\mathcal{CL} \Rightarrow \mathcal{JS}$

- Let the Laplace transform, defined for $\Re(\omega) > 0$

$$\tilde{a}(\omega) = \int_0^{\infty} a(x)e^{-\omega x} dx$$

– We have

$$f(z)e^z = \frac{1}{2i\pi} \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} \tilde{a}(\omega) \exp(ze^{\omega}) d\omega$$



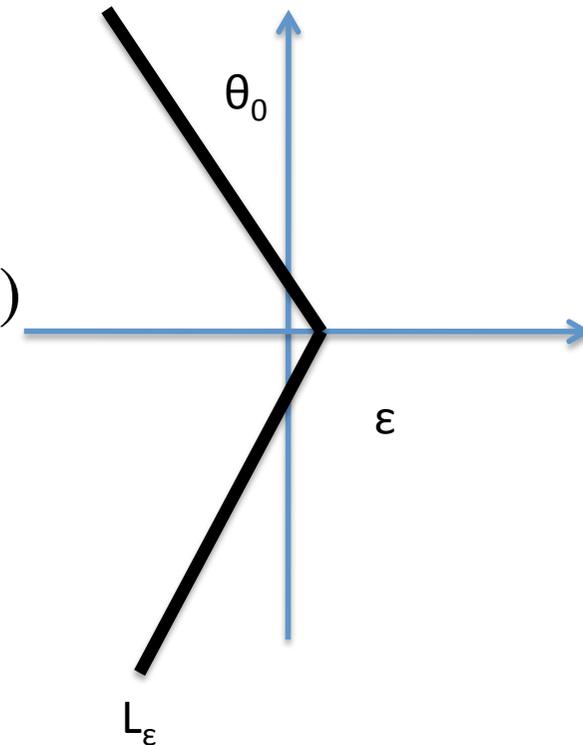
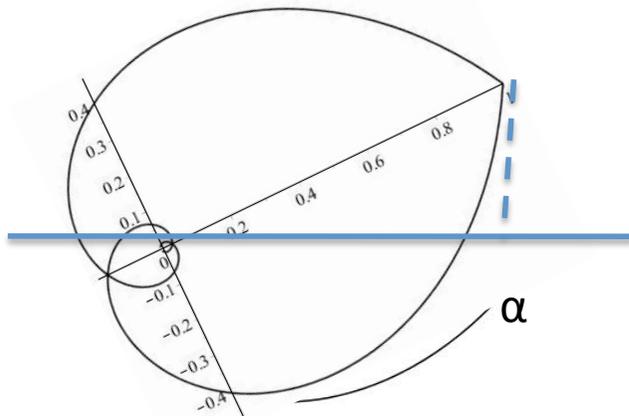
Proof $\mathcal{CL} \Rightarrow \mathcal{JS}$ Condition (O)

- If $a(x)$ is x^β in a cone of angle θ_0 we have

$$a(x) = \frac{1}{2i\pi} \int_{L_\varepsilon} \tilde{a}(\omega) \exp(x\omega) d\omega$$

$$f(z)e^z = \frac{1}{2i\pi} \int_{L_\varepsilon} \tilde{a}(\omega) \exp(ze^\omega) d\omega$$

– If $\Re(ze^\omega) \leq \alpha|z| \Rightarrow f(z)e^z = O(e^{\alpha|z|})$



Proof $\mathcal{CL} \Rightarrow \mathcal{JS}$ Condition (I)

- Complex z in a cone

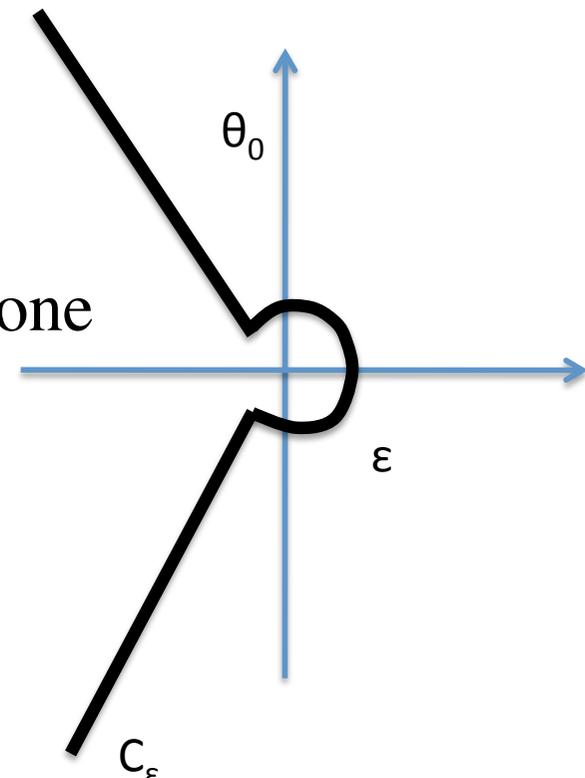
$$f(z) = \frac{1}{2i\pi} \int_{C_\varepsilon} \tilde{a}(\omega) \exp((e^\omega - 1)z) d\omega$$

– We have

$$\tilde{a}(\omega) = O(\omega^{-\beta-1}), \text{ when } \omega \rightarrow 0 \text{ in a cone}$$

– Estimate like in Flajolet Odlyzko

$$f(z) = O(z^\beta)$$



Flajolet, P., & Odlyzko, A. (1990). Singularity analysis of generating functions. *SIAM Journal on discrete mathematics*, 3(2), 216-240.

Conclusion and perspective

- Extension to multiple dePoissonization seems easy
- Work to do for
 - diagonal dePoissonization
 - Exponential dePoissonization
- More work to do for
 - Polynomial cone