## The Quintet

# Poisson-Mellin-Newton-Rice-Laplace 

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## Plan of the talk

Two probabilistic models, the Bernoulli model and the Poisson model.
with their tools, the Poisson transform, the Poisson sequence.

Two paths from the Poisson model to the Bernoulli model

- The first path: Depoissonization path with the Poisson transform.
- The second path: Newton-Rice path with the Poisson sequence.

Here, in this talk:

- Survey of the two paths
- Study of the Rice path using the Laplace transform
- Comparison of the two paths.
- It is true the Rice path be more restrictive to use?


## Plan of the talk

I. General framework.
II. The Depoissonization path
III. The Rice path
IV. The Rice-Laplace path
V. Comparaison between the two paths
I. General framework.

## Two probabilistic models.

Space of inputs := the set $\mathcal{X}^{\star}$ of the finite sequences $x$ of elements of $\mathcal{X}$.
There are two main probabilistic models on the set $\mathcal{X}^{\star}$.

- The Bernoulli model $\mathcal{B}_{n}$, where the cardinality $N$ of the sequence $\boldsymbol{x}$ is fixed to $n$ (then $n \rightarrow \infty$ );
The Bernoulli model is more natural in algorithmics.
- The Poisson model $\mathcal{P}_{z}$ of parameter $z$, where the cardinality $N$ of the sequence $\boldsymbol{x}$ is a random variable that follows a Poisson law of parameter $z$,

$$
\operatorname{Pr}[N=n]=e^{-z} \frac{z^{n}}{n!},
$$

(then $z \rightarrow \infty$ ). The Poisson model has nice probabilistic properties, notably independence properties $\Longrightarrow$ easier to deal with.
$\Longrightarrow A$ first study in the Poisson model, followed with a return to the Bernoulli model

## Average-case analysis of a cost $R$ defined on $\mathcal{X}^{\star}$ $\mathcal{X}^{\star}:=$ set of the finite sequences of elements of $\mathcal{X}$

- Final aim : Study the sequence $n \mapsto f(n)$,
$f(n):=\mathbb{E}_{[n]}[R]:=$ the expectation in the Bernoulli model $\mathcal{B}_{n}$
- Consider the expectation $\mathbb{E}_{z}[R]$ in the Poisson model $\mathcal{P}_{z}$

$$
\begin{aligned}
\mathbb{E}_{z}[R] & =\sum_{n \geq 0} \mathbb{E}_{z}[R \mid N=n] \mathbb{P}_{z}[N=n] \\
& =\sum_{n \geq 0} \mathbb{E}_{[n]}[R] \mathbb{P}_{z}[N=n]=e^{-z} \sum_{n \geq 0} f(n) \frac{z^{n}}{n!}
\end{aligned}
$$

$\mathbb{E}_{z}[R]$ is the Poisson transform $P_{f}(z)$ of the sequence $n \mapsto f(n)$.

- With (properties of) the Poisson transform $P_{f}(z)$ of $f: n \mapsto f(n)$ return to (the asymptotics of) the sequence $n \mapsto f(n)$

With a sequence $f: n \mapsto f(n)$ [the expectations in the $\mathcal{B}_{n}$ models],
we associate $\quad P(z)=e^{-z} \sum_{k \geq 0} f(k) \frac{z^{k}}{k!}=\sum_{k \geq 0}(-1)^{k} \frac{z^{k}}{k!} p(k)$

- The series $P(z):=P_{f}(z)$ is the Poisson transform of $n \mapsto f(n)$, also the expectation in the Poisson model $\mathcal{P}_{z}$
- The sequence $k \mapsto p(k)$ is the Poisson sequence of $n \mapsto f(n)$. It is denoted by $\Pi[f]$. The map $\Pi$ is involutive. Important involutive binomial relation between $f(n)$ and $p(n)$

$$
p(n)=\sum_{k=0}^{n}(-1)^{k}\binom{n}{k} f(k), \quad \text { and } \quad f(n)=\sum_{k=0}^{n}(-1)^{k}\binom{n}{k} p(k) .
$$

Begin with a sequence $k \mapsto f(k)$ of polynomial growth, consider its Poisson transform $P(z)$ and its Poisson sequence $\Pi[f]: n \mapsto p(n)$,

$$
P(z)=e^{-z} \sum_{k \geq 0} f(k) \frac{z^{k}}{k!}=\sum_{n \geq 0}(-1)^{n} \frac{z^{n}}{n!} p(n)
$$

Assume some "knowledge"
on the Poisson transform $P_{f}(z)$ or on the Poisson sequence $\Pi[f]$.
There are two paths for returning to the asymptotics of the initial sequence

- Depoissonisation method:
- Deal with $P(z)$, find its asymptotics $(z \rightarrow \infty)$ [tools à la Mellin]
- Compare the asymptotics of the sequence $f(n)(n \rightarrow \infty)$ to the asymptotics of $P(n)$
- Rice method
- Deal with the sequence $\Pi[f]: n \mapsto p(n)$,
- and its analytic lifting $\psi$ which exists [tools à la Mellin-Rice].
- Return to the sequence $n \mapsto f(n)$ via the binomial formula which is tranfered into the Rice integral.

An instance of application: Toll functions and tries (I).

A source $\mathcal{S}$ on a finite alphabet $\Sigma$
$\mathcal{X}^{\star}:=\left\{\right.$ sequences of words of $\Sigma^{\mathbb{N}}$ produced by a source $\left.\mathcal{S}\right\}$

The trie $\mathcal{T}(\boldsymbol{x})$ built on $\boldsymbol{x} \in \mathcal{X}^{\star}$ is a tree :

- If $|\boldsymbol{x}|=0, \mathcal{T}(\boldsymbol{x})=\emptyset$
- If $|\boldsymbol{x}|=1, \mathcal{T}(\boldsymbol{x})$ is a leaf
- If $|x| \geq 2$, then $\mathcal{T}(\boldsymbol{x})$ is formed with
- an internal node o
- and a sequence of tries $\mathcal{T}\left(\boldsymbol{x}_{\sigma}\right)$ for $\sigma \in \Sigma$

- Each internal node is labelled with a prefix $\boldsymbol{w}$,
- The associated subtrie deals with the words of $\boldsymbol{x}$ which begin with $\boldsymbol{w}$.

A sequence $n \mapsto f(n)$ with $\operatorname{val}(f) \geq 2$ plays the role of a toll function.

With the toll $f$, associate the cost $R$ defined on $\mathcal{X}^{\star}$

$$
R(\boldsymbol{x}):=\sum_{\boldsymbol{w} \in \Sigma^{\star}} f\left(N_{\boldsymbol{w}}(\boldsymbol{x})\right),
$$

- $N_{\boldsymbol{w}}(\boldsymbol{x})$ is the number of words of $\boldsymbol{x}$ which begin with the prefix $\boldsymbol{w}$
- $f\left(N_{\boldsymbol{w}}(\boldsymbol{x})\right)$ is the toll "paid"

by the subtrie whose root is labelled by $\boldsymbol{w}$
$f(k)=1 \Longrightarrow R(\boldsymbol{x})$ is the number of internal nodes of $\mathcal{T}(\boldsymbol{x})$
$f(k)=k \Longrightarrow R(\boldsymbol{x})$ is the external path length of $\mathcal{T}(\boldsymbol{x})$
Another instance (less classical) : $f(k)=k \log k \Longrightarrow \ldots .$.
$R(\boldsymbol{x})$ is the number of symbol comparisons performed by QuickSort on $\boldsymbol{x}$.
What is the mean value of the cost $R(\boldsymbol{x})$ when $\boldsymbol{x} \in \mathcal{X}^{n}$ ?

An instance of application: Toll functions and tries (III).
Study $r(n):=$ the mean value of $R:=\sum_{\boldsymbol{w} \in \Sigma^{\star}} f\left(N_{\boldsymbol{w}}\right)$ in the $\mathcal{B}_{n}$ model
The toll $f$ gives the Poisson transform $P_{f}(z)$ and/or its sequence $\pi[f]$

$$
\left.P_{f}(z)=\mathbb{E}_{z}[f(N))\right]=e^{-z} \sum_{n \geq 2} f(n) \frac{z^{n}}{n!}=\sum_{n \geq 2}(-1)^{n} p(n) \frac{z^{n}}{n!}
$$

$N$ follows $\mathcal{P}_{z} \Longrightarrow N_{\boldsymbol{w}}$ follows $\mathcal{P}_{z \pi_{w}} \Longrightarrow \mathbb{E}_{z}\left[f\left(N_{\boldsymbol{w}}\right)\right]=P\left(z \pi_{\boldsymbol{w}}\right)$
For $\boldsymbol{w} \in \Sigma^{\star}, \pi_{\boldsymbol{w}}:=$ the probability that a word begins with $\boldsymbol{w}$

$$
\begin{array}{r}
Q_{R}(z):=\mathbb{E}_{z}[R]=\sum_{w \in \Sigma^{\star}} \mathbb{E}_{z}\left[f\left(N_{\boldsymbol{w}}\right)\right] \Longrightarrow Q_{R}(z)=\sum_{w \in \Sigma^{\star}} P\left(z \pi_{\boldsymbol{w}}\right) \\
Q_{R}(z)=e^{-z} \sum_{n \geq 2} r(n) \frac{z^{n}}{n!}=\sum_{n \geq 2}(-1)^{n} q(n) \frac{z^{n}}{n!} \Longrightarrow q(n)=\left[\sum_{w \in \Sigma^{\star}} \pi_{w}^{n}\right] p(n)
\end{array}
$$

Sequence $f(n) \Longrightarrow$ A good knowledge of $P_{f}(z)$ and/or $p(n)$ With the $\left(\pi_{\boldsymbol{w}}\right)$ of source $\mathcal{S} \Longrightarrow$ A good knowledge of $Q(z)$ and/or $q(n)$ How to return to $r(n)$ ?

II - The Depoissonization path

The Depoissonization path deals with the Poisson transform $P(z)$. It

- compares $f(n)$ and $P(n)$ with the Poisson-Charlier expansion
- uses the Mellin inverse transform for the asymptotics of $P(n)$
- needs depoissonization sufficient conditions $\mathcal{J S}$, for truncating the Poisson-Charlier expansion
- obtains the asymptotics of $f(n)$.
- better understands the $\mathcal{J S}$ conditions: they are true in any practical situation !

Main contributors

- Haymann [1956]
- Jacquet and Szpankowski [1998] (two papers), Jacquet [2014]
- Hwang-Fuchs-Zacharovas [2010]


## The Depoissonization path (I). The Charlier-Poisson expansion

 introduced in the AofA domain by Hwang-Fuchs-Zacharovas [2010]$$
\text { Taylor expansion at } z=n: \quad P(z)=\sum_{j \geq 0} \frac{P^{(j)}(n)}{j!}(z-n)^{j} .
$$

As $P(z)$ is entire, there is an infinite expansion of $f(n)$ in terms of $P^{(j)}(n)$

$$
f(n):=n!\left[z^{n}\right]\left(e^{z} P(z)\right)=\sum_{j \geq 0} \frac{P^{(j)}(n)}{j!} \tau_{j}(n),
$$

with the Charlier-Poisson polynomials $n \mapsto \tau_{j}(n):=n!\left[z^{n}\right]\left((z-n)^{j} e^{z}\right)$,
This infinite expansion is always valid. But we wish truncate ...
What happens when we only keep the first terms? Which error is expected?
We need here Depoissonization conditions on the Poisson transform $P(z) \ldots$.

## The Depoissonization path (II). $\mathcal{J S}$ Conditions for depoissonisation

There are sufficient conditions on cones provided by Haymann (1956), and introduced in the AofA domain by Jacquet and Szpankowski (1998)
[ $\mathcal{J S}$ admissibility] An entire function $P(z)$ is $\mathcal{J S}$-admissible with parameters $(\alpha, \beta)$ if there exist $\theta \in] 0, \pi / 2[, \delta<1$ for which (for $z \rightarrow \infty$ ) (I) For $\arg z \leq \theta$, one has $|P(z)|=O\left(|z|^{\alpha} \log ^{\beta}(1+|z|)\right)$.
( $O$ ) For $\theta \leq \arg z \leq \pi$, one has $\left|P(z) e^{z}\right|=O\left(e^{\delta|z|)}\right)$.

Theorem. (Jacquet-Szpankowski[1998] Hwang-Fuchs-Zacharovas[2010]) If the Poisson transform $P(z)$ of $f(n)$ is $\mathcal{J} \mathcal{S}(\alpha, \beta)$ admissible, then

$$
f(n)=\sum_{0 \leq j<2 k} P^{(j)}(n) \frac{\tau_{j}(n)}{j!}+O\left(n^{\alpha-k} \log ^{\beta} n\right)
$$

## The Depoissonization path (III) : The main result.

Yes! But which conditions on the initial sequence $f$ itself?

Theorem. (Jacquet and Szpankowski [1998], Jacquet [2014])
Let $P(z)$ be the Poisson transform of $f(n)$ assumed to be entire.

- The two conditions are equivalent
(i) $P(z)$ is $\mathcal{J S}$-admissible
(ii) The sequence $n \mapsto f(n)$ admits an analytical lifting $\varphi(z)$
on the half plane $\Re s>-1$
of polynomial growth in a cone $\mathcal{C}\left(-1, \theta_{0}\right)$ for some $\theta_{0}>0$.

Theorem. Assume that the sequence $n \mapsto f(n)$ admits an analytical lifting $\varphi(z)$ on the half plane $\Re s>-1$ of polynomial growth in a cone $\mathcal{C}\left(-1, \theta_{0}\right)$ for some $\theta_{0}>0$.
Then the truncation of the Poisson-Charlier expansion gives rise to an estimate of the sequence $n \mapsto f(n)$ with "good" remainder terms.

III - The Rice path

For a sequence $n \mapsto f(n)$ of polynomial growth, the Rice path deals with the Poisson sequence $\Pi[f]$.

- It proves the existence of an analytical lifting $\psi$ of the sequence $\Pi[f]$ with the (direct) Mellin transform and Newton interpolation. without any other condition on the sequence $n \mapsto f(n)$.
- If moreover $\psi$ is of polynomial growth, the binomial relation is transfered into a Rice integral expression
- With a shifting to the left, it provides the asymptotics of $f(n)$.

Main contributors

- Norlünd, Norlünd-Rice
- Flajolet and Sedgewick [1995]

What are the sufficient conditions for polynomial growth of $\psi$ ?
Not well studied! Are they true in any practical situation ?

The main object of this talk.

## The Rice path (0). Technical conditions.

## Valuation-Degree Condition

Definition. For a non zero real sequence $n \mapsto f(n)$ of polynomial growth,

$$
\operatorname{deg}(f):=\inf \left\{c \mid f(k)=O\left(k^{c}\right)\right\} \quad \operatorname{val}(f):=\min \{k \mid f(k) \neq 0\},
$$

The sequence $n \mapsto f(n)$ satisfies the Valuation-Degree Condition (VD),

$$
\text { iff } \operatorname{val}(f)>\operatorname{deg}(f)+1
$$

An important condition, but easy to ensure:
Main interest in the asymptotics of $f: n \mapsto f(n)$
$\Longrightarrow$ we put $k_{0}$ zeroes (with $k_{0}>d+1$ ) at the beginning of $n \mapsto f(n)$ to obtain a sequence $f_{+}$of valuation $k_{0}>d+1$, which satisfies the VD condition.

$$
f_{+}(n)=0 \text { for } n<k_{0}, \quad f_{+}(n)=f(n) \text { for } n \geq k_{0}
$$

The Rice path (0). Technical conditions.
An important tool: Shifting the sequences.
When $\operatorname{val}\left(f_{+}\right)=k_{0}>d+1, P(z)$ is written as

$$
P(z)=z^{k_{0}} Q(z), \quad Q(z)=e^{-z} \sum_{k \geq 0} g(k) \frac{z^{k}}{k!}=\sum_{k \geq 0}(-1)^{k} \frac{z^{k}}{k!} q(k) .
$$

The "shifted" sequences $g$ and $q:=\Pi[g]$ are expressed with the "shifting" map $T$ and its $\ell$-iterates

$$
T[f](n)=\frac{f(n+1)}{n+1}, \quad T^{\ell}[f](n)=\frac{f(n+\ell)}{(n+1) \ldots(n+\ell)} .
$$

Remark that the shifting $T$ "almost" commutes with the involution $\Pi$

$$
T \circ \Pi=-\Pi \circ T, \quad \Pi \circ T^{\ell}=(-1)^{\ell} T^{\ell} \circ \Pi
$$

One has: $\quad g=T^{k_{0}}\left[f_{+}\right], \quad q:=\Pi[g]=(-1)^{k_{0}} T^{k_{0}}\left[\Pi\left[f_{+}\right]\right]$ and also: $\quad \operatorname{val}(g)=0, \quad \operatorname{deg}(g)=d-k_{0}<-1$

Conclusion. We will perform the study for the sequences $g$ and $q$. Then we easily return (via $T^{-k_{0}}$ ) to the initial sequences $f_{+}$and $\Pi\left[f_{+}\right]$ It is then enough to deal with $\operatorname{val}(g)=0, \quad \operatorname{deg}(g):=c<-1$

## Mellin-Newton: Analytic lifting $\psi$ of $\Pi[f]$

Proposition. [Nordlünd-Rice] For a sequence $n \mapsto f(n)$ with $\operatorname{val}(f)=0$, $\operatorname{deg}(f)=c<0$, the sequence $\Pi[f]$ admits as an analytic lifting the function $\psi$ on $\Re s>c$

$$
\psi(s)=\sum_{k \geq 0}(-1)^{k} f(k) \frac{s(s-1) \ldots(s-k+1)}{k!}
$$

which is also an analytic extension of $P^{*}(-s) / \Gamma(-s)$.

Proof. In the strip $\langle 0,-c\rangle$, the Mellin transform $P^{*}(s)$ of $P(z)$ exists and

$$
\frac{P^{*}(s)}{\Gamma(s)}=\frac{1}{\Gamma(s)} \sum_{k \geq 0} \frac{f(k)}{k!} \int_{0}^{\infty} e^{-z} z^{k} z^{s-1} d z=\sum_{k \geq 0} \frac{f(k)}{k!} \frac{\Gamma(k+s)}{\Gamma(s)}
$$

(The exchange of integration and summation is justified)

On the strip $\langle c, 0\rangle$, the right series is a Newton interpolation series...

$$
\psi(s):=\frac{P^{*}(-s)}{\Gamma(-s)}=\sum_{k \geq 0}(-1)^{k} f(k) \frac{s(s-1) \ldots(s-k+1)}{k!}
$$

which converges in right halfplanes and thus on $\Re s>c$.
Moreover,
$\psi(n)=\sum_{k=0}^{n}(-1)^{k} f(k) \frac{n(n-1) \ldots(n-k+1)}{k!}=\sum_{k=0}^{n}(-1)^{k}\binom{n}{k} f(k)=\Pi[f](n)$

This proves that there is an analytic lifting $\psi(s)$ of $n \mapsto \Pi[f](n)$ on $\Re s>c$ which is also an analytic extension of $P^{*}(-s) / \Gamma(-s)$.

## Shifting to the right - Rice transform

The binomial relation between $f(n)$ and $\Pi[f](n)$ is transfered into a Rice integral.
Assume that the analytic lifting $\psi$ of $\Pi[f]$ is of polynomial growth as $s \rightarrow \infty$ on $\Re s>c$. Then, for any $a \in] c, 0\left[\right.$ and $n \geq n_{0}$, one has:

$$
f(n)=\sum_{k=0}^{n}(-1)^{k}\binom{n}{k} p(k) \Longrightarrow f(n)=\frac{1}{2 i \pi} \int_{a-i \infty}^{a+i \infty} L_{n}(s) \cdot \psi(s) d s
$$

with the Rice kernel

$$
L_{n}(s)=\frac{(-1)^{n+1} n!}{s(s-1)(s-2) \ldots(s-n)}=\frac{\Gamma(n+1) \Gamma(-s)}{\Gamma(n+1-s)}=B(n+1,-s) .
$$

Proof: Residue Theorem; uses the polynomial growth of $\psi(s)$ to the right.
This integral representation is valid for $a \in[c, 0]$.
We now shift to the left $\ldots$ and we again need tameness conditions on $\psi$,

## The Rice path (III).

Tameness of $\psi$ and shifting to the left.
Definition. A function $\varpi$ analytic and of polynomial growth on $\Re s>c$ is tame at $s=c$ if there exists a region $\mathcal{R}$ between a curve $\mathcal{C} \subset\{\Re s<c\}$ and $\Re s=c$ for which $\varpi$ is meromorphic and of polynomial growth on $\mathcal{R}$.

Proposition. Consider $n \mapsto f(n)$ with $\operatorname{val}(f)=0$ and $\operatorname{deg} f=c<0$. If the lifting $\psi$ of $\Pi[f]$ is tame at $s=c$, then

$$
f(n)=-\left[\sum_{k \mid s_{k} \in \mathcal{R}} \operatorname{Res}\left[L_{n}(s) \cdot \psi(s) ; s=s_{k}\right]+\frac{1}{2 i \pi} \int_{\mathcal{C}} L_{n}(s) \cdot \psi d s\right],
$$

The sum is over the poles $s_{k}$ of $\psi$ inside $\mathcal{R}$.
Often easy to apply $\ldots$ but we need tameness of $\psi(s)=\frac{P^{*}(-s)}{\Gamma(-s)}$.

## The Rice path (IV).

## Sufficient conditions for tameness of $\psi$ ?

$$
\psi(s)=\frac{P^{*}(-s)}{\Gamma(-s)}=\sum_{k \geq 0}(-1)^{k} f(k) \frac{s(s-1) \ldots(s-k+1)}{k!} .
$$

Closely related to the Mellin transform $P^{*}(s)$.
Meromorphy is often easy to ensure, the poles are easy to find...
And polynomial growth? True for $P^{*}(-s)$. But with the factor $1 / \Gamma(-s)$ ?
Sometimes [often in tries problems], the factor $\Gamma(s)$ appears in $P^{*}(s)$.
But what about other sequences, for instance $f(k)=k \log k$

- where the depoissonization path can be used.
- Is the Rice path useful in this case?
- Is it true that the Rice path is useful only for very specific cases?

We now propose to use the Laplace (inverse) transform.....

## IV - The Rice-Laplace approach.

$$
P(z)=e^{-z} \sum_{n \geq 0} f(n) \frac{z^{n}}{n!}=\sum_{n \geq 0}(-1)^{n} p(n) \frac{z^{n}}{n!}
$$

Recall the involution $\Pi$ between the sequences $n \mapsto p(n)$ and $n \mapsto f(n) \ldots$

## Proposition.

(a) Assume that the Poisson sequence $n \mapsto p(n)$ admits an analytic lifting $\psi$ on $\Re s>-1$ which is of polynomial growth on each $\Re s>a$ for $a>-1$. Then, the sequence $n \mapsto f(n)$ admits an analytic lifting $\varphi$ on $\Re s>-1$ which is of polynomial growth on each $\Re s>a$ for $a>-1$.
(b) Assume that the sequence $n \mapsto f(n)$ admits an analytic lifting $\varphi$ on $\Re s>-1$ which is of polynomial growth on each $\Re s>a$ for $a>-1$. Then, the sequence $n \mapsto p(n)$ admits an analytic lifting $\psi$ on $\Re s>-1$ which is of polynomial growth on each $\Re s>a$ for $a>-1$

The Rice-Laplace approach (I): Use the inverse Laplace transform.
We prove $(b)$. With shifting $n \mapsto f(n)$, we may assume $\operatorname{deg}(f)<-1$.
Proposition. Consider a sequence $f: n \mapsto f(n)$ which
(a) admits an analytic lifting $\varphi$ on $\Re s>a$ with $a \in]-1,0[$,
(b) with the estimate $\varphi(s)=O\left(|s+1|^{c}\right)$ there with $c<-1$.

## Then:

(i) The function $\varphi$ admits an inverse Laplace transform $\widehat{\varphi}$ whose restriction to the real line $[0,+\infty[$ is written as the Bromwich integral for $b \in] a, 0[$,

$$
\widehat{\varphi}(u)=\frac{1}{2 i \pi} \int_{\Re s=b} \varphi(s) e^{s u} d s, \quad \text { and satisfies }|\widehat{\varphi}(u)| \leq K e^{b u}
$$

(ii) The analytical lifting $\psi$ of the sequence $\Pi[f]$ exists on $\Re s>-1$, is expressed as an integral on the real line,

$$
\psi(s)=\int_{0}^{+\infty} \widehat{\varphi}(u) \cdot\left(1-e^{-u}\right)^{s} d u
$$

Compare with the expression $\varphi(s)=\int_{0}^{+\infty} \widehat{\varphi}(u) \cdot e^{-s u} d u$

Proposition. Consider a sequence $f: n \mapsto f(n)$ which
(a) admits an analytic lifting $\varphi$ on $\Re s>a$ with $a \in]-1,0[$,
(b) with the estimate $\varphi(s)=O\left(|s+1|^{c}\right)$ there with $c<-1$.

Then, there is an analytical lifting $\psi$ of the sequence $\Pi[f]$ on $\Re t>-1$ that is expressed as an integral on the real line,

$$
\psi(t)=\int_{0}^{+\infty} \widehat{\varphi}(u) \cdot\left(1-e^{-u}\right)^{t} d u
$$

## Proof.

Transfer the binomial expression of $\Pi[f]$ in terms of $f$ into a Rice integral,

$$
p(n)=\frac{1}{2 i \pi} \int_{\Re s=b} \varphi(s) L_{n}(s) d s, \quad L_{n}(s)=\frac{\Gamma(n+1) \Gamma(-s)}{\Gamma(n+1-s)}, \quad b<0 .
$$

Extend $L_{n}(s)=B(n+1,-s)$ with the Beta function, and use the integral expression

$$
B(t+1,-s)=\int_{0}^{\infty} e^{s u}\left(1-e^{-u}\right)^{t} d u, \quad \Re t>-1, \Re s<0
$$

With properties of $\varphi$, interverting the integrals
$\psi(t)=\frac{1}{2 i \pi} \int_{\Re s=b} \varphi(s) B(t+1,-s) d s=\int_{0}^{\infty}\left(1-e^{-u}\right)^{t}\left[\frac{1}{2 i \pi} \int_{\Re s=b} \varphi(s) e^{s u} d s\right] d u$

## The Rice-Laplace approach (II).

Proposition. Consider a sequence $f: n \mapsto f(n)$ which

- admits an analytic lifting $\varphi$ on $\Re s>a$ with $a \in]-1,0[$,
- with the estimate $\varphi(s)=O\left(|s+1|^{c}\right)$ there with $c<-1$.

Then: there is an integral form for the analytical lifting $\psi$ of the $\Pi[f]$ sequence

$$
\psi(s)=\int_{0}^{+\infty} \widehat{\varphi}(u) \cdot\left(1-e^{-u}\right)^{s} d u, \quad \text { for } \Re s>-1
$$

(i) A good expansion of $u \mapsto \widehat{\varphi}(u)$ may easily prove the tameness of $s \mapsto \psi(s)$ on the left of $\Re s=-1$
(ii) With Rice's principles, shifting to the left the integral

$$
f(n)=\frac{1}{2 i \pi} \int_{a-i \infty}^{a+i \infty} L_{n}(s) \cdot \psi(s) d s
$$

provides the asymptotics of the sequence $n \mapsto f(n)$.

## The Rice-Laplace approach (III).

A particular class of interest: Basic functions.
Definition. Consider a triple $\left(k_{0}, d, b\right)$ with

$$
\text { a real } d \text {, two integers } b \geq 0, k_{0}:=\operatorname{Val}(f), k_{0}>\max (d+1,1) \text {, }
$$

A sequence $k \mapsto f(k)$ is called basic with the triple $\left(k_{0}, d, b\right)$ if

$$
f(k)=k^{d} \log ^{b} k S(1 / k) \quad \text { for } k \geq k_{0}, \quad f(k)=0 \quad \text { for } k<k_{0}
$$

$S$ is analytic at 0 with a convergence radius $r=1 /\left(k_{0}-1\right)$, and $S(0) \neq 0$.

The canonical sequence $g: T^{k_{0}}[f]$ is extended into $\left.\left.g:\right]-1,+\infty\right] \rightarrow \mathbb{R}$

$$
g(x)=\left(x+k_{0}\right)^{c} \log ^{b}\left(x+k_{0}\right) U\left(1 /\left(x+k_{0}\right)\right), \quad c=d-k_{0}<-1
$$

$U$ is analytic at 0 with a convergence radius $r=1 /\left(k_{0}-1\right)$, and $U(0) \neq 0$.

Ex: $f(k)=k \log k$ with $k_{0}=3 \Longrightarrow g(x)=\frac{\log (x+3)}{(x+1)(x+2)}=\frac{\log (x+3)}{(x+3)^{2}} U\left(\frac{1}{x+3}\right)$

The Rice-Laplace approach (IV).
Tameness of the analytical lifting for basic functions.

Proposition. Consider the canonical sequence $g(n)$ associated with a basic sequence of the form $f(k)=k^{d} \log ^{b} k S(1 / k)$. Let $c=d-k_{0}<-1$. Then:

- Its inverse Laplace transform $\widehat{g}$ is a linear combination of functions

$$
e^{-k_{0} u} u^{-c-1}\left(\log ^{\ell} u\right) V_{\ell}(u) \quad \text { for } \ell \in[0 . . b]
$$

where the functions $V_{\ell}$ satisfy $V_{\ell}(0) \neq 0$ and $\left|V_{\ell}(u)\right| \leq e^{(u / 2)\left(2 k_{0}-1\right)}$

- The analytical lifting $\psi$ of $\Pi[g]$ is a linear combination of functions

$$
\int_{0}^{\infty} e^{-k_{0} u} u^{-c-1+s}\left(\log ^{\ell} u\right) V_{\ell}(u)\left(\frac{1-e^{-u}}{u}\right)^{s} d u \quad \text { for } \ell \in[0 . . b]
$$

- $s \mapsto \psi(s)$ is meromorphic on $\Re s>c-1$,
- with an only pole at $s=c$ of multiplicity $b+1$,
- it is of polynomial growth in any half-plane $\Re s \geq \sigma_{0}>c-1$.
V. Comparison between the two paths: Depoissonization and Rice-Laplace

The Rice-Laplace method deals with sequences $f$
that have analytic lifting of polynomial growth on half-planes.

- It exhibits an integral representation of the analytical lifting $\psi$ of $\Pi[f]$
- proves the tameness of $\psi$ for basic sequences, notably $f(k)=k \log k$

This validates the Rice path for such sequences
in the same framework as Depoissonization, but more restricted.
If the sequence $f$ satisfies the VD condition and the $\mathcal{J S}$ condition, then:

- the canonical sequence $g=T^{k_{0}}[f]$
- admits an analytical lifting $g(z)$ in any cone $\mathcal{C}(-1, \theta)$
- which is of polynomial growth of degree $c<-1$ in a cone $\mathcal{C}\left(-1, \theta_{0}\right)$.
- If moreover, the angle $\theta_{0}$ satisfies $\theta_{0} \geq \pi / 2$, then the Rice-Laplace path may be used.

Is it possible to extend the Rice method for an angle $\theta_{0}<\pi / 2$ ?
Is it possible to use the Laplace (inverse) transform in this case?

